MVC Revision Checklist (and references to problem sheet questions)

1) • Defn of a map. Linearity: defn and examples; ps1q1

$\mathbb{R}^m \ni \underline{x} \to \underline{F}(\underline{x}) \in \mathbb{R}^n$ (vector)

$\underline{x} \to F(\underline{x})$ (scalar)

linear: $\underline{F}(\mu \underline{x} + \lambda \underline{y}) = \mu \underline{F}(\underline{x}) + \lambda \underline{F}(\underline{y})$

or $\exists A$ s.t. $\underline{F} = A\underline{x}$

constant.

2) • Derivative of a map (Jacobian matrix); ps1q3, ps1q3, ps1q8

$$\underline{F}'(\underline{x}) = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_m} \\ \\ \frac{\partial F_1}{\partial x_m} & & \frac{\partial F_n}{\partial x_m} \end{pmatrix}$$

3) • Directional derivatives: both ways of defining; ps1q3

$$D_{\underline{v}} \underline{F}(\underline{x}_0) = \underline{F}'(\underline{x}_0)\underline{v} \quad \text{or} \quad = \frac{d}{dt}\left(\underline{F}(\underline{x}_0 + t\underline{v})\right)_{t=0}$$

Note $|\underline{v}| = 1$ (normalise)

4) • Chain rule applied to "functions of functions"; ps1q2, ps1q4

if $\underline{H}(\underline{x}) = (\underline{G} \circ \underline{F})(\underline{x}) = \underline{G}(\underline{F}(\underline{x}))$ then

$$\underline{H}'(\underline{x}) = \underline{G}'(\underline{F}(\underline{x}))\underline{F}'(\underline{x})$$

5) • Defn of inverse map & its derivation (derivative of inverse = inverse of derivative); ps3q1

$(\underline{F}^{-1} \circ \underline{F}) = \underline{I}$ where $\underline{x} = \underline{I}\underline{x}$.

$\Rightarrow (\underline{F}^{-1})' = (\underline{F}')^{-1}$ from (4)

6) • Inverse function theorem and application to existence solutions of systems of equations; p1q5, p1q8

if $\underline{y}_0 = \underline{f}(\underline{x}_0)$ & $\underline{F}'(\underline{x}_0)^*$ (is invertible) then

$\underline{y} = \underline{F}(\underline{x})$ has sol^{n}s (i.e. $\underline{x} = \underline{F}^{-1}(\underline{y})$) for $\underline{y}$ in nhd of $\underline{y}_0$

7) • Implicit function theorem and application to existence of solutions of systems of equations; ps1q7

(*) $J_{\underline{F}}(\underline{x}_0) = \left.\frac{\partial(F_1, \ldots F_n)}{\partial(x_1, \ldots, x_n)}\right|_{\underline{x}=\underline{x}_0} \neq 0$. Jacobian determinant

if $\underline{F}(\underline{x}_0, \underline{y}_0) = \underline{0}$ & $\left.\frac{\partial(F_1, \ldots, F_n)}{\partial(y_1, \ldots, y_n)}\right|_{\underline{x}=\underline{x}_0, \underline{y}=\underline{y}_0} \neq 0$ then $\exists$ sol^n

of $\underline{F}(\underline{x}, \underline{y}) = 0$ in nbhd of $\underline{x}_0, \underline{y}_0$.

i.e. $\underline{y} = \underline{y}(\underline{x})$ for some $\underline{G}$

- Higher order derivatives; Taylor's theorem; ps1q6

Scalar: $f(\underline{x}) = f(\underline{x}_0) + (\underline{x}-\underline{x}_0).\nabla f + \frac{1}{2}(\underline{x}-\underline{x}_0)^T \left(\underset{\uparrow}{H} \right)_{\underline{x}_0} (\underline{x}-\underline{x}_0) + \ldots$

Hessian

vector: $\underline{F}(\underline{x}) = \underline{f}(\underline{x}_0) + \underline{F}'(\underline{x}_0)(\underline{x}-\underline{x}_0) + \ldots$

- Summation convention (pairs of suffices imply summation); ps2q1

$$u_i v_i \equiv \sum_{i=1}^{3} u_i v_i = \underline{u}.\underline{v}$$

- Kronecker delta defn and rule of application; ps2q1, ps2q4, ps2q5, ps2q6, ps2q7, ps2q8

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{not} \end{cases} \qquad x_j \delta_{ij} = x_i$$

- Levi-civita tensor defn and rule of application to cross products; ps2q4, ps2q5, ps2q6, ps2q7, ps2q8

$$[\underline{u} \times \underline{v}]_i = \varepsilon_{ijk} u_j v_k$$

$\uparrow$ i^{th} component.

$\boxed{\varepsilon_{123} = 1}$

$\varepsilon_{ijk} = 0$ if two suffices the same

$\varepsilon_{ijk} = -\varepsilon_{jik}$ (interchange)

$\varepsilon_{jki} = \varepsilon_{ijk}$. (rotation)

- Connection between products of e_{ijk} and products of delta_{ij}; ps2q5,ps2q7,ps2q8

$$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

- Defn of gradient and interpretations; ps2q3, ps2q4, ps2q6

for f scalar ∇f is <u>gradient</u> $= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

<u>or</u> $[\nabla f]_i = \frac{\partial f}{\partial x_i}$

- ∇f point in dirn of steepest ascent
- ∇f is perpendicular to level curves (f = constant) of f.

- Defn of divergence and its application; ps2q3, ps2q4, ps2q5, ps2q6, ps2q7, ps2q8

$$\nabla \cdot \underline{v} = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} = \frac{\partial v_i}{\partial x_i}$$

Note $\underline{v} \cdot \nabla = v_i \frac{\partial}{\partial x_i} \neq \frac{\partial}{\partial x_i} v_i$

- Defn of curl and its application; ps2q3, ps2q4, ps2q5, ps2q6, ps2q7, ps2q8

$$[\nabla \times \underline{v}]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} v_k = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial_x & \partial_y & \partial_z \\ v_1 & v_2 & v_3 \end{vmatrix}$$

- The two Null identities; ps2q3, ps2q10

  - $\nabla \times \nabla f = 0$ for $\underline{\text{all}}$ scalar f
  - $\nabla \cdot \nabla \times \underline{v} = 0$ for all vector $\underline{v}$

  + $\underline{\text{proofs}}$

- Defn of the Laplacian; ps2q4, ps2q6, ps2q10, ps3q3, ps3q5

$$\nabla^2 f \equiv \Delta f = \nabla \cdot \nabla f$$

also applies to vector fields $\underline{v}$ so $\nabla^2 \underline{v}$ is a thing

- Coordinate transformations, the defn of a curvilinear basis, scale factors, the right-hand rule for ordering of basis, invertibility of the map; ps3q2, ps3p4

if $\underline{r} = \underline{r}(\underline{q})$ is a map from $\underline{q} = (q_1, q_2, q_3)$ system to $\underline{r} = (x, y, z)$ then $\hat{q}_\alpha = \frac{1}{h_\alpha} \frac{\partial \underline{r}}{\partial q_\alpha}$ & $h_\alpha = \left| \frac{\partial \underline{r}}{\partial q_\alpha} \right|$ is scale factor

are basis vectors in new system.

- Transformation of the gradient: method and result; ps3q2, ps3p4

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \hat{\underline{x}} \frac{\partial}{\partial x} + \hat{\underline{y}} \frac{\partial}{\partial y} + \hat{\underline{z}} \frac{\partial}{\partial z} = \sum_{\alpha=1}^{3} \frac{1}{h_\alpha} \hat{\underline{q}}_\alpha \frac{\partial}{\partial q_\alpha} = \frac{\hat{\underline{q}}_1}{h_1} \frac{\partial}{\partial q_1} + \frac{\hat{\underline{q}}_2}{h_2} \frac{\partial}{\partial q_2} + \frac{\hat{\underline{q}}_3}{h_3} \frac{\partial}{\partial q_3}$$

- Transformation of the divergence (awareness of how this is done, not to remember derivation or formulae); ps3q2, ps3p4

– key points are product rules: $\nabla \cdot (f \underline{v}) = f \nabla \cdot \underline{v} + \nabla f \cdot \underline{v}$

& $\nabla \times (f \underline{v}) = f \nabla \times \underline{v} + \nabla f \times \underline{v}$.

– Also e.g. $\frac{\partial}{\partial \theta} (\phi \hat{\underline{\theta}}) = \hat{\underline{\theta}} \frac{\partial \phi}{\partial \theta} + \phi \frac{\partial \hat{\underline{\theta}}}{\partial \theta}$

- Line integrals of scalar and vector fields in 3D space: definitions, parametrisation of curves; ps4q1, ps4q2, ps4q4, ps4q5,. ps4q6

◦ Scalar line integral: $\int_C f(\underline{r})\,ds = \int_{t_1}^{t_2} f(\underline{r}(t))|\underline{r}'(t)|\,dt$

where C is parametrised by $\underline{r}(t)$, $t_1 < t < t_2$ & $ds = |d\underline{r}|$

• vector line integral: $\int_C \underline{V}(\underline{r}).d\underline{r} = \int_{t_1}^{t_2} \underline{V}(\underline{r}(t)).\underline{r}'(t)\,dt$

- Fundamental theorem of Calculus for line integrals: proof and consequences; ps4q7, ps4q8, ps5q2

$$\int_C \nabla f.d\underline{r} = f(\underline{r}(t_2)) - f(\underline{r}(t_1)) \leftarrow \text{proof by chain rule}$$

$$= 0 \text{ if } C \text{ is } \underline{\text{closed}}$$

- Surface integrals of scalar and vector fields: definitions, how to parametrise surfaces, notion of surface direction; ps4q3, ps4q5, ps4q6

• Scalar surface integral: $\int_S f(\underline{r})\,dS = \iint_D f(\underline{s}(u,v))|\underline{N}|\,du\,dv$

where $\underline{s}(u,v)$ parametrises S over $(u,v) \in D$ & $dS = |d\underline{S}|$

→→ • vector surface integral: $\int_S \underline{V}(\underline{r}).d\underline{S} = \iint_D \underline{V}(\underline{s}(u,v)).\underline{N}\,du\,dv$ ⊛

- Stokes' theorem: definition, the RH-thumb rule for consistent orientation of surfaces and curves (awareness of method of proof, but not details); ps4q5, ps4q6, ps5q1

$$\int_S \nabla \times \underline{A}.d\underline{S} = \int_{\partial S} \underline{A}.d\underline{r}$$

for $\underline{A}$ a vector field & ∂S is bdy. of S

dir^n of S and orientation of path integral ∂S must adhere to RH thumb rule.

- Green's theorem in the plane: defn and application; ps4q7, ps5q2, ps5q3.

if $\underline{A} = (A_1, A_2, 0)$ & $A_i = A_i(x,y)$

then

$$\int_S \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) dS = \int_{\partial S} \underline{A}.d\underline{r} = \int_S A_1\,dx + A_2\,dy.$$

- Volume integrals and change of coordinates; ps5q4, ps5q5, ps5q6, ps5q8

$$\int_V f(\underline{r})\,dx\,dy\,dz = \int_{V'} f(\underline{r}(\underline{q}))|J_r|\,dq_1\,dq_2\,dq_3$$

where V' is mapped to V by $\underline{r}(\underline{q})$

⊛ dir^n of $d\underline{S} = \underline{N}\,du\,dv$ important

$\underline{N} = \underline{s}_u \times \underline{s}_v$ points normal to S

- Divergence theorem: defn (awareness of proof, but not details) and application; ps5q4, ps5q5, ps5q6, ps5q8

$$\int_V \nabla \cdot \underline{F}\, dV = \int_{\partial V} \underline{F} \cdot d\underline{S}$$

$\rightarrow$ dir$^{\underline{n}}$ of $d\underline{S}$ must be out of V

- Green's Identities: defn and derivation from the divergence theorem; ps5q10.

- $$\int_V \Delta f \, dV = \int_{\partial V} \hat{\underline{n}} \cdot \nabla f \, dS$$

- $$\int_V (g \Delta f - f \Delta g)\, dV = \int_{\partial V} (g\, \hat{\underline{n}} \cdot \nabla f - f\, \hat{\underline{n}} \cdot \nabla g)\, dS$$