MATH20901 Multivariable Calculus: Solutions 1 [1]

1. (i) Is a linear map, and can easily be seen to satisfy the requirement of a linear map that $\mathbf{F}(\lambda\mathbf{x}+\mu\mathbf{y}) = \lambda\mathbf{F}(\mathbf{x})+\mu\mathbf{F}(\mathbf{y})$. Equivalently, we can find a matrix A s.t. $\mathbf{F}(\mathbf{x}) = A\mathbf{x}$ and here

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

(ii) Satisfies $\mathbf{F}(\lambda\mathbf{x}) = \lambda^2\mathbf{F}(\mathbf{x}) \neq \lambda\mathbf{F}(\mathbf{x})$ and is therefore not linear.

(iii) As in (i), linear, and

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

2. $(\mathbf{G}\circ\mathbf{F})(\mathbf{x}) = \mathbf{G}(-x_2, x_1) = (x_1, -\sin x_2)$ and $(\mathbf{F}\circ\mathbf{G})(\mathbf{x}) = \mathbf{F}(x_2, \sin x_1) = (-\sin x_1, x_2)$.

3. (a) Simple matter of computing the partial derivatives. The matrix $\mathbf{F}'(\mathbf{x})$ is

$$\begin{pmatrix} \dfrac{\partial F_1}{\partial x_1} & \dfrac{\partial F_1}{\partial x_2} \\ \dfrac{\partial F_2}{\partial x_1} & \dfrac{\partial F_2}{\partial x_2} \\ \dfrac{\partial F_3}{\partial x_1} & \dfrac{\partial F_3}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1x_2 & x_1^2 \\ \cos(x_1+x_2) & \cos(x_1+x_2) \\ x_2e^{x_1x_2} & x_1e^{x_1x_2} \end{pmatrix}$$

(b) From the definition, remembering we need to normalise $\mathbf{v}$ so $\hat{\mathbf{v}} = (1,2)/\sqrt{5}$ and

$$\begin{aligned} D_{\mathbf{v}}\mathbf{F}(\mathbf{x}) &= \frac{d}{dt}\left((x_1+t/\sqrt{5})^2(x_2+2t/\sqrt{5}), \sin(x_1+x_2+3t/\sqrt{5}), e^{(x_1+t/\sqrt{5})(x_2+2t/\sqrt{5})}\right)\Big|_{t=0} \\ &= \frac{1}{\sqrt{5}}\left(2x_1x_2+2x_1^2, 3\cos(x_1+x_2), (x_2+2x_1)e^{x_1x_2}\right). \end{aligned}$$

(c) Using $\mathbf{x} = (1,1)$ in part (b) gives $D_{\mathbf{v}}\mathbf{F}(\mathbf{x}) = (4, 3\cos 2, 3e)/\sqrt{5}$, while $\mathbf{x} = (1,1)$ in (a) gives

$$\mathbf{F}'(1,1)\hat{\mathbf{v}} = \begin{pmatrix} 2 & 1 \\ \cos 2 & \cos 2 \\ e & e \end{pmatrix}\begin{pmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{pmatrix} = \begin{pmatrix} 4/\sqrt{5} \\ (3\cos 2)/\sqrt{5} \\ 3e/\sqrt{5} \end{pmatrix}$$

The two results agree, as required.

4. (a) From the Chain rule (see notes),

$$\mathbf{H}'(\mathbf{x}) = \mathbf{G}'(\mathbf{F}(\mathbf{x}))\mathbf{F}'(\mathbf{x}).$$

Thus,

$$\mathbf{H}'(1,1) = \mathbf{G}'(\mathbf{F}(1,1))\mathbf{F}'(1,1).$$

We have that (see notes)

$$\mathbf{F}'(\mathbf{x}) = A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Also,

$$\mathbf{F}(1,1) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 1 \end{pmatrix}.$$

Now we have $\mathbf{G}(\mathbf{x}) = (x_1x_2, x_2x_3, \sin(x_1x_2x_3)) \equiv (G_1, G_2, G_3)$ so

$$\begin{aligned}
\mathbf{G}'(\mathbf{x}) &\equiv \begin{pmatrix} \frac{\partial G_1}{\partial x_1} & \frac{\partial G_1}{\partial x_2} & \frac{\partial G_1}{\partial x_3} \\ \frac{\partial G_2}{\partial x_1} & \frac{\partial G_2}{\partial x_2} & \frac{\partial G_2}{\partial x_3} \\ \frac{\partial G_3}{\partial x_1} & \frac{\partial G_3}{\partial x_2} & \frac{\partial G_3}{\partial x_3} \end{pmatrix} \\
&= \begin{pmatrix} x_2 & x_1 & 0 \\ 0 & x_3 & x_2 \\ x_2x_3\cos(x_1x_2x_3) & x_3x_1\cos(x_1x_2x_3) & x_1x_2\cos(x_1x_2x_3) \end{pmatrix}.
\end{aligned}$$

OK, so we already have $\mathbf{F}(1,1) = (3,3,1)$, so we get

$$\mathbf{G}'(\mathbf{F}(1,1)) = \mathbf{G}'(3,3,1) = \begin{pmatrix} 3 & 3 & 0 \\ 0 & 1 & 3 \\ 3\cos 9 & 3\cos 9 & 9\cos 9 \end{pmatrix}.$$

Then

$$\begin{aligned}
\mathbf{H}'(1,1) &= \mathbf{G}'(\mathbf{F}(1,1))\mathbf{F}'(1,1) \\
&= \begin{pmatrix} 3 & 3 & 0 \\ 0 & 1 & 3 \\ 3\cos 9 & 3\cos 9 & 9\cos 9 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 9 & 9 \\ 5 & 1 \\ 18\cos 9 & 9\cos 9 \end{pmatrix}.
\end{aligned}$$

(b) We have that, for $\mathbf{x} = (x_1, x_2)$,

$$\mathbf{F}(\mathbf{x}) = A\mathbf{x} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ 2x_1 + x_2 \\ x_1 \end{pmatrix}$$

Then

$$\begin{aligned}
\mathbf{H}(\mathbf{x}) &= \mathbf{G}(\mathbf{F}(\mathbf{x})) = \mathbf{G}(x_1 + 2x_2, 2x_1 + x_2, x_1) \\
&= ((x_1 + 2x_2)(2x_1 + x_2), (2x_1 + x_2)x_1, \sin((x_1 + 2x_2)(2x_1 + x_2)x_1))
\end{aligned}$$

and we write $\mathbf{H} \equiv (H_1, H_2, H_3)$. Then

$$\mathbf{H}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} \\ \frac{\partial H_3}{\partial x_1} & \frac{\partial H_3}{\partial x_2} \end{pmatrix}$$

$$= \begin{pmatrix} 4x_1 + 5x_2 & 4x_2 + 5x_1 \\ 4x_1 + x_2 & x_1 \\ (6x_1^2 + 10x_1x_2 + 2x_2^2)\times & (4x_2x_1 + 5x_1^2)\times \\ \cos((x_1 + 2x_2)(2x_1 + x_2)x_1) & \cos((x_1 + 2x_2)(2x_1 + x_2)x_1) \end{pmatrix}.$$

Urgh ! Now we put in $\mathbf{x} = (1, 1)$ and we find the same answer as at the end of part (a).

5. Here, $\mathbf{F}(x, y) = (x^3 + e^y, \cos x + xy)$ and so

$$\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} 3x^2 & e^y \\ -\sin x + y & x \end{pmatrix}$$

is the Jacobian matrix. Its determinant is just

$$J_{\mathbf{F}} = 3x^3 + e^y(\sin x - y)$$

and this clearly vanishes where $(x, y) = (0, 0)$. So the relation $\mathbf{F}(\mathbf{x}) = \mathbf{s}$ where $\mathbf{s} = (s, t)$ is not invertible, according to the notes at $(x, y) = (0, 0)$ and a unique solution is therefore not guaranteed there.

6. Use Taylor's theorem (notes). So first $\mathbf{F}(1, 2) = (5, 9)$. Next,

$$\mathbf{F}'(\mathbf{x}) = \begin{pmatrix} 2x & 2y \\ 1 - y^3/x^2 & 3y^2/x \end{pmatrix}$$

So

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}(1, 2) + \mathbf{F}'(1, 2)(\mathbf{x} - (1, 2)) + \text{h.o.t.}$$

if $\mathbf{x}$ is close to $(1, 2)$ the higher order terms are small and so

$$\mathbf{F}(\mathbf{x}) \approx (5, 9)^T + \begin{pmatrix} 2 & 4 \\ -7 & 12 \end{pmatrix} \begin{pmatrix} x - 1 \\ y - 2 \end{pmatrix}$$

and then you're almost there.

7. Call the first equation $F_1(x, y, u, v) = 0$ and the second $F_2(x, y, u, v) = 0$. For the system to be uniquely determined near a point the determinant of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix} = \begin{pmatrix} -2uy & 2v \\ \frac{yv}{u^2} - 3 & -\frac{y}{u} \end{pmatrix}$$

evaluated at $(x, y, u, v) = (2, 1, 1, -1)$ must be non-vanishing. Using these values in the above gives a determinant of -6.

Why is this ? Well, since $u = u(x, y)$ and $v = v(x, y)$ by the chain rule we have for example

$$\frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F_1}{\partial v}\frac{\partial v}{\partial x} = 0$$

and so on. The four equations that result can be arranged as the matrix equation

$$\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} + \begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = 0$$

and this gives

$$\begin{pmatrix} 2x & 2y - u^2 \\ y^2 & 2xy - \frac{v}{u} \end{pmatrix} + \begin{pmatrix} -2uy & 2v \\ \frac{yv}{u^2} - 3 & -\frac{y}{u} \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = 0$$

so evaluating these at $(x, y, u, v) = (2, 1, 1, -1)$ and inverting to get the derivatives requires the determinant of the Jacobian previous computed to be non-zero. If we do this numerical task we find the unknown $\partial v/\partial y = -1$.

8. (a) First

$$\mathbf{r}'(r, \phi, \theta) = \begin{pmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0. \end{pmatrix}$$

(b) The Jacobian is

$$\begin{aligned} J_{\mathbf{r}} \equiv \det(\mathbf{r}') \equiv \frac{\partial(x, y, z)}{\partial(r, \phi, \theta)} &= \begin{vmatrix} \sin\phi\cos\theta & r\cos\phi\cos\theta & -r\sin\phi\sin\theta \\ \sin\phi\sin\theta & r\cos\phi\sin\theta & r\sin\phi\cos\theta \\ \cos\phi & -r\sin\phi & 0 \end{vmatrix} \\ &= \ldots = r^2\sin\phi. \end{aligned}$$

We can solve (r, ϕ, θ) in terms of (x, y, z) everywhere except where the Jacobian determinant vanishes. This happens when $\sin\phi = 0$ or when $\phi = 0$, $\phi = \pi$ which are the polar axes, or $\mathbf{r} = (0, 0, \pm r)$ which is the z-axis.

9. (a) First, clear to see that f is continuous if $\mathbf{x} \neq 0$, since the numerator and denominator are both continuous and the denominator is non-vanishing.

(b) Let the path $x = y^3$ be parametrised by $x = t^3$, $y = t$. Then

$$\lim_{t \to 0} f(\mathbf{x}(t)) = \frac{t^6}{t^6 + t^6} = \frac{1}{2}$$

and this isn't the same as $f(0, 0) = 0$. So discontinuous.