# MATH20901 Multivariable Calculus: Solutions 2 [1]

1. (a) $\delta_{ij}\delta_{jk}\delta_{ki} = \delta_{ij}\delta_{ji} = \delta_{jj} = 3$.

   $\epsilon_{ijk}\epsilon_{ijk}$ is the sum of the squares of all 27 values of $\epsilon_{ijk}$. So the answer is 6. Or you could answer using the double product result given in the notes:

   $$\epsilon_{ijk}\epsilon_{ijk} = \delta_{jj}\delta_{kk} - \delta_{jk}\delta_{kj} = 3.3 - \delta_{jj} = 9 - 3 = 6.$$

   (b) $(AB^TC)_{ij} = A_{il}B_{kl}C_{kj} \equiv \sum_{l=1}^{p}\sum_{k=1}^{q} A_{il}B^T_{lk}C_{kj}$ and $B^T_{lk} = B_{kl}$, observing that $A$ has $p$ columns, and $C$ has $q$ rows.

2. Part (i) has a geometric interpretation: $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is the volume of the parallelepiped with edges formed by the vectors $\mathbf{a},\mathbf{b},\mathbf{c}$. As such $\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$ is the same volume, although the ordering of the vectors is important as the result can be negated if the ordering of the cross-product is reversed – so it's not quite so trivial. Mathematically, we can do this:

   $$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = a_i\epsilon_{ijk}b_jc_k = b_j\epsilon_{jki}c_ka_i = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a})$$

   after rotating the suffices of $\epsilon_{ijk}$ and ordering the elements of the vectors in the correct manner.

   For part (ii), we have to prove a vector identity and so we consider the $i$th component:

   $$\begin{aligned}[(\mathbf{a}\times\mathbf{b})\times(\mathbf{a}\times\mathbf{c})]_i = \epsilon_{ijk}[\mathbf{a}\times\mathbf{b}]_j[\mathbf{a}\times\mathbf{c}]_k &= \epsilon_{ijk}\epsilon_{jlm}a_lb_m\epsilon_{krs}a_rc_s \\ &= \epsilon_{jki}\epsilon_{jlm}a_lb_m\epsilon_{krs}a_rc_s \\ &= (\delta_{kl}\delta_{im} - \delta_{km}\delta_{il})a_lb_m\epsilon_{krs}a_rc_s \\ &= (a_kb_i\epsilon_{krs}a_rc_s) - (a_ib_k\epsilon_{krs}a_rc_s) \\ &= b_i(\mathbf{a}\cdot(\mathbf{a}\times\mathbf{c})) - a_i(\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}))\end{aligned}$$

   The first term on the right-hand side is zero (standard result for vectors) and the second is the $i$th component of $-(\mathbf{b}\cdot(\mathbf{a}\times\mathbf{c}))\,\mathbf{a}$, which we can write as $(\mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}))\,\mathbf{a}$.

3. (a) $\nabla f(\mathbf{r}) = (-y\sin(xy), -x\sin(xy) - z\sin(yz), -y\sin(yz))$. So

   $$\begin{aligned}\nabla\times\nabla f &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ -y\sin(xy) & -x\sin(xy) - z\sin(yz) & -y\sin(yz)\end{vmatrix} \\ &= (-\sin(yz) - yz\cos(yz) + \sin(yz) + yz\cos(yz))\hat{\mathbf{x}} + \ldots = 0\end{aligned}$$

   Has to be so, as proved in notes for any $f$.

   (b) $\nabla\cdot\mathbf{u} = \sin z + z - \sin z = z$.

   (c)

   $$\nabla\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ ayz & bzx & cxy\end{vmatrix} = ((c-b)x, (a-c)y, (b-a)z).$$

   Then $\nabla\cdot(\nabla\times\mathbf{v}) = (c-b) + (a-c) + (b-a) = 0$ as required from the proof in the notes.

4. (a) Here $f = \mathbf{a}\cdot\mathbf{r}$ and $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{r} = (x_1, x_2, x_3)$ so $f = a_j x_j$ and $[\boldsymbol{\nabla} f]_i = \dfrac{\partial}{\partial x_i}(a_j x_j) = a_j \delta_{ij} = a_i$; thus $\boldsymbol{\nabla} f(\mathbf{r}) = \mathbf{a}$.

(b) First, $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ so $\boldsymbol{\nabla} r = (x, y, z)/r = \mathbf{r}/r$. Next

$$\mathbf{v} = \boldsymbol{\nabla} r^n = n r^{n-1} \boldsymbol{\nabla} r = n r^{n-2} \mathbf{r}$$

Continuing, we have

$$\boldsymbol{\nabla}\cdot\mathbf{v} = \frac{\partial v_i}{\partial x_i} = \frac{\partial}{\partial x_i}(n x_i r^{n-2}) = n\frac{\partial x_i}{\partial x_i} r^{n-2} + n x_i \left((n-2) x_i r^{n-4}\right) = 3n r^{n-2} + n(n-2) r^2 r^{n-4}$$

after using the first differentiation result again for the second term of the product. So

$$\boldsymbol{\nabla}\cdot\mathbf{v} = n(n+1) r^{n-2}$$

which vanishes for $n = 0$ and $n = -1$, provided $r \neq 0$. We had to expect that $n = 0$ was one solution as $r^0 = 1$.

(c) Here $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ and $\mathbf{r} = (x_1, x_2, x_3)$. The $i$th component of the curl is

$$\begin{aligned}[\boldsymbol{\nabla}\times\mathbf{v}]_i &= \epsilon_{ijk}\frac{\partial}{\partial x_j} v_k = \epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{klm}\omega_l x_m = \epsilon_{ijk}\epsilon_{klm}\omega_l\frac{\partial x_m}{\partial x_j} = \epsilon_{ijk}\epsilon_{klm}\omega_l\delta_{jm} = \epsilon_{imk}\epsilon_{klm}\omega_l \\ &= \epsilon_{kim}\epsilon_{klm}\omega_l = (\delta_{il}\delta_{mm} - \delta_{im}\delta_{ml})\omega_l = (3\delta_{il} - \delta_{il})\omega_l = 2\omega_i.\end{aligned}$$

So we have $\boldsymbol{\nabla}\times\mathbf{v} = 2\boldsymbol{\omega}$.

5. (a)(i) Similar to 4(c) above, but we also have $\boldsymbol{\nabla} r = \mathbf{r}/r$ or

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

The $i$ component is

$$\begin{aligned}[\boldsymbol{\nabla}\times(\mathbf{r}\times\mathbf{a}f(r))]_i &= \epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{klm} x_l a_m f(r) = \epsilon_{kij}\epsilon_{klm}\left(\frac{\partial x_l}{\partial x_j} a_m f(r) + x_l a_m \frac{x_j}{r} f'(r)\right) \\ &= \epsilon_{kij}\epsilon_{klm}\left(\delta_{lj} a_m f(r) + a_m \frac{x_l x_j}{r} f'(r)\right) \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\left(\delta_{lj} a_m f(r) + a_m \frac{x_l x_j}{r} f'(r)\right) \\ &= a_i f(r) - \delta_{jj} a_i f(r) + a_j \frac{x_i x_j}{r} f'(r) - a_i \frac{x_j^2}{r} f' \\ &= \left[-\mathbf{a}(2f(r) + r f'(r)) + \mathbf{r}\frac{\mathbf{a}\cdot\mathbf{r}}{r} f'(r)\right]_i\end{aligned}$$

and so

$$\boldsymbol{\nabla}\times(\mathbf{r}\times\mathbf{a}f(r)) = -\mathbf{a}(2f(r) + r f'(r)) + \mathbf{r}\frac{\mathbf{a}\cdot\mathbf{r}}{r} f'(r).$$

(a)(ii) Same tricks as above, slightly easier now

$$\boldsymbol{\nabla}\cdot\mathbf{a}f(r) = \frac{\partial}{\partial x_i}(a_i f(r)) = a_i\frac{\partial f}{\partial x_i} = a_i\frac{x_i}{r} f'(r)$$

using (a)(i). So $\boldsymbol{\nabla}\cdot\mathbf{a}f(r)=\dfrac{\mathbf{a}\cdot\mathbf{r}}{r}f'(r)$.

(b)

$$\begin{aligned}[\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u})]_i &= \epsilon_{ijk}u_j\epsilon_{klm}\frac{\partial}{\partial x_l}u_m=(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\,u_j\frac{\partial}{\partial x_l}u_m \\ &= u_m\frac{\partial}{\partial x_i}u_m-u_l\frac{\partial}{\partial x_l}u_i=\frac{\partial}{\partial x_i}\left(\tfrac{1}{2}u_m^2\right)-u_l\frac{\partial}{\partial x_l}u_i=\left(\tfrac{1}{2}\boldsymbol{\nabla}\mathbf{u}^2-[\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right]_i,\end{aligned}$$

which is the $i$th component of $\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u})=\frac{1}{2}\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{u})-(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}$.

6. (a) $\boldsymbol{\nabla}\cdot(f\mathbf{v})=\dfrac{\partial}{\partial x_i}(fv_i)=f\dfrac{\partial v_i}{\partial x_i}+v_i\dfrac{\partial f}{\partial x_i}=f\boldsymbol{\nabla}\cdot\mathbf{v}+\mathbf{v}\cdot\boldsymbol{\nabla}f$

   (b) This is just part (a) with $\mathbf{v}=\boldsymbol{\nabla}g$ and the only thing to note here is that $\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}g=\Delta g$, the Laplacian of $g$.

   (c) Take the $i$th component of the LHS:

$$[\boldsymbol{\nabla}\times(f\mathbf{v})]_i=\epsilon_{ijk}\frac{\partial}{\partial x_j}(fv_k)=\epsilon_{ijk}f\frac{\partial v_k}{\partial x_j}+\epsilon_{ijk}\frac{\partial f}{\partial x_j}v_k=f[\boldsymbol{\nabla}\times\mathbf{v}]_i+[\boldsymbol{\nabla}f\times\mathbf{v}]_i$$

7. On the LHS if you switch over $\mathbf{u}$ and $\mathbf{v}$, by the definition of the cross product you will introduce a minus sign. However the RHS is symmetric in $\mathbf{u}$ and $\mathbf{v}$ and so switching them over will give the same result. So it cannot be true as stated. Here's the derivation.

$$\begin{aligned}\boldsymbol{\nabla}\cdot(\mathbf{u}\times\mathbf{v}) &= \frac{\partial}{\partial x_i}(\epsilon_{ijk}u_jv_k)=\epsilon_{ijk}\left(u_j\frac{\partial v_k}{\partial x_i}+v_k\frac{\partial u_j}{\partial x_i}\right) \\ &= -u_j\epsilon_{jik}\frac{\partial v_k}{\partial x_i}+v_k\epsilon_{kij}\frac{\partial u_j}{\partial x_i} \\ &= -\mathbf{u}\cdot(\boldsymbol{\nabla}\times\mathbf{v})+\mathbf{v}\cdot(\boldsymbol{\nabla}\times\mathbf{u})\end{aligned}$$

   In the above we have used the cyclic definition of $\epsilon_{ijk}$.

8. Similar to above

$$\begin{aligned}[\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{v})]_i &= \epsilon_{ijk}\frac{\partial}{\partial x_j}(\epsilon_{klm}u_lv_m)=\epsilon_{kij}\epsilon_{klm}\left(u_l\frac{\partial v_m}{\partial x_j}+v_m\frac{\partial u_l}{\partial x_j}\right) \\ &= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\left(u_l\frac{\partial v_m}{\partial x_j}+v_m\frac{\partial u_l}{\partial x_j}\right)=u_i\frac{\partial v_j}{\partial x_j}+v_j\frac{\partial u_i}{\partial x_j}-u_j\frac{\partial v_i}{\partial x_j}-v_i\frac{\partial u_j}{\partial x_j} \\ &= (\boldsymbol{\nabla}\cdot\mathbf{v})u_i+(\mathbf{v}\cdot\boldsymbol{\nabla})u_i-(\mathbf{u}\cdot\boldsymbol{\nabla})v_i-(\boldsymbol{\nabla}\cdot\mathbf{u})v_i\end{aligned}$$

   So we match up each suffix to give the vector result

9. (a) Here $\mathbf{F}=-\rho g\hat{\mathbf{z}}=\boldsymbol{\nabla}\phi$. So

$$\phi_x=0,\quad \phi_y=0,\quad \phi_z=-\rho g$$

   Integrating each up gives $\phi(x,y,z)=f_1(y,z)$, $\phi(x,y,z)=f_2(x,z)$ and $\phi(x,y,z)=-\rho gz+f_3(x,y)$ where $f_1,f_2,f_3$ are arbitrary functions. In fact, the only way this can resolve itself is if $f_1=f_2=f_3=C$ a constant. So, in general $\phi=-\rho gz+C$.

(b) You do this in Fluids 3. We have

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla} p$$

after using part (a). Using Q3(b) we have

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u}) = -\frac{1}{\rho}\boldsymbol{\nabla} p$$

Taking the curl, noting that $\boldsymbol{\nabla}\times\boldsymbol{\nabla} f = 0$ for any scalar $f$, and defining $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$ we have

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{\nabla}\times(\mathbf{u}\times\boldsymbol{\omega}) = 0$$

Now using Q6, we can write

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - (\boldsymbol{\nabla}\cdot\boldsymbol{\omega})\mathbf{u} + (\boldsymbol{\nabla}\cdot\mathbf{u})\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} = 0$$

Finally, since $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ and $\boldsymbol{\nabla}\cdot\boldsymbol{\omega} = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\mathbf{u} = 0$, we can write

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{u}$$

(c) If $\mathbf{u} = (u_1(x,y), u_2(x,y), 0)$, then $\boldsymbol{\omega} = (\partial_x u_2 - \partial_y u_1)\hat{\mathbf{z}}$ and both $(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{u} = 0$ and $(\mathbf{u}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} = 0$. Which means that $\partial_t\boldsymbol{\omega} = 0$ and so $\boldsymbol{\omega}$ is constant.

10. Using result (i) in the question we can write Navier's equation as

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \mu\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{u})$$

Now taking the divergence of this and using result (iii) on the last term to eliminate it gives

$$\rho\frac{\partial^2}{\partial t^2}(\boldsymbol{\nabla}\cdot\mathbf{u}) = (\lambda + 2\mu)\Delta(\boldsymbol{\nabla}\cdot\mathbf{u})$$

and letting $\phi = \boldsymbol{\nabla}\cdot\mathbf{u}$ and $c_1^2 = (\lambda + 2\mu)/\rho$ we have the set equation.

Now take the curl and let $\mathbf{H} = \boldsymbol{\nabla}\times\mathbf{u}$ we can eliminate the first term on the RHS to leave

$$\rho\frac{\partial^2}{\partial t^2}\mathbf{H} = \mu\boldsymbol{\nabla}\times\Delta\mathbf{u} = -\mu\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{H})$$

once results (i), (ii) are used. We need one more result, given in §2.3.1 of the notes

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{H}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{H}) - \Delta\mathbf{H})$$

and since $\mathbf{H} = \boldsymbol{\nabla}\times\mathbf{u}$ the first term is zero. Hence we have

$$\rho\frac{\partial^2}{\partial t^2}\mathbf{H} = \mu\Delta\mathbf{H}$$

as required with $c_2^2 = \mu/\rho$.

The reduction of Navier's equation to these two decoupled equations is very important in the study of Seismology as they represent *wave equations* for the dilation (or compressible) and rotational components of displacements in a solid. The factors $c_1$ and $c_2$ are *wave speeds* (see APDE2) and clearly $c_1 > c_2$. This means compression waves travel faster than rotational waves. In Siesmology the two waves and called P and S waves – the P is for primary (because they arrive first) and the S for secondary.