# MATH20901 Multivariable Calculus: Solutions 3 [1]

1. (a) From the notes the definition of the derivative of the map $\mathbf{r}(\mathbf{q})$ is

$$\mathbf{r}'(\mathbf{q}) = \begin{pmatrix} \dfrac{\partial x}{\partial q_1} & \dfrac{\partial x}{\partial q_2} & \dfrac{\partial x}{\partial q_3} \\ \dfrac{\partial y}{\partial q_1} & \dfrac{\partial y}{\partial q_2} & \dfrac{\partial y}{\partial q_3} \\ \dfrac{\partial z}{\partial q_1} & \dfrac{\partial z}{\partial q_2} & \dfrac{\partial z}{\partial q_3} \end{pmatrix} \equiv \begin{pmatrix} [\hat{\mathbf{q}}_1]_1 & [\hat{\mathbf{q}}_2]_1 & [\hat{\mathbf{q}}_3]_1 \\ [\hat{\mathbf{q}}_1]_2 & [\hat{\mathbf{q}}_2]_2 & [\hat{\mathbf{q}}_3]_2 \\ [\hat{\mathbf{q}}_1]_3 & [\hat{\mathbf{q}}_2]_3 & [\hat{\mathbf{q}}_3]_3 \end{pmatrix} \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix},$$

where $[\hat{\mathbf{q}}_\alpha]_i$ is the $i$th component of $\alpha$th basis vector. This uses $\hat{\mathbf{q}}_\alpha = (1/h_\alpha)\partial\mathbf{r}/\partial q_\alpha$ where $\mathbf{r} = (x, y, z)$.

(b) Note that $(AB)^{-1} = B^{-1}A^{-1}$ and that the inverse of a diagonal matrix is the matrix of reciprocals on the diagonal. Also, since the $\hat{\mathbf{q}}$ are normalised and orthogonal to one another, the matrix made up of them is orthonormal and therefore its inverse is equal to its transpose. Thus

$$(\mathbf{r}'(\mathbf{q}))^{-1} = \begin{pmatrix} h_1^{-1} & 0 & 0 \\ 0 & h_2^{-1} & 0 \\ 0 & 0 & h_3^{-1} \end{pmatrix} \begin{pmatrix} [\hat{\mathbf{q}}_1]_1 & [\hat{\mathbf{q}}_1]_2 & [\hat{\mathbf{q}}_1]_3 \\ [\hat{\mathbf{q}}_2]_1 & [\hat{\mathbf{q}}_2]_2 & [\hat{\mathbf{q}}_2]_3 \\ [\hat{\mathbf{q}}_3]_1 & [\hat{\mathbf{q}}_3]_2 & [\hat{\mathbf{q}}_3]_3 \end{pmatrix} = \begin{pmatrix} [\hat{\mathbf{q}}_1]_1/h_1 & [\hat{\mathbf{q}}_1]_2/h_1 & [\hat{\mathbf{q}}_1]_3/h_1 \\ [\hat{\mathbf{q}}_2]_1/h_2 & [\hat{\mathbf{q}}_2]_2/h_2 & [\hat{\mathbf{q}}_2]_3/h_2 \\ [\hat{\mathbf{q}}_3]_1/h_3 & [\hat{\mathbf{q}}_3]_2/h_3 & [\hat{\mathbf{q}}_3]_3/h_3 \end{pmatrix}.$$

The result follows after using $\hat{\mathbf{q}}_\alpha = (1/h_\alpha)\partial\mathbf{r}/\partial q_\alpha$ again

(c) The first thing to note is that $(\mathbf{r}'(\mathbf{q}))^{-1} = \mathbf{q}'(\mathbf{r})$ (the inverse of the derivative is the derivative of the inverse and here $\mathbf{q}(\mathbf{r})$ denotes the inverse map), from the notes on inverse maps in Chapter 1. Which means to say that

$$\mathbf{q}'(\mathbf{r}) = \begin{pmatrix} \dfrac{\partial q_1}{\partial x} & \dfrac{\partial q_1}{\partial y} & \dfrac{\partial q_1}{\partial z} \\ \dfrac{\partial q_2}{\partial x} & \dfrac{\partial q_2}{\partial y} & \dfrac{\partial q_2}{\partial z} \\ \dfrac{\partial q_3}{\partial x} & \dfrac{\partial q_3}{\partial y} & \dfrac{\partial q_3}{\partial z} \end{pmatrix} = (\mathbf{r}'(\mathbf{q}))^{-1}.$$

In spherical coordinates, $\mathbf{q} = (r, \phi, \theta)$ and so $q_2 \equiv \phi$ and we therefore want the $(2, 2)$ entry of the matrix above which is the same as the $(2, 2)$ entry of the matrix in part (b). Hence

$$\frac{\partial\phi}{\partial y} = \frac{1}{h_\phi^2}\frac{\partial y}{\partial\phi} = \frac{r\cos\phi\sin\theta}{r^2} = \frac{\cos\phi\sin\theta}{r}$$

using the definition of the map in spherical polars, $y = r\sin\phi\sin\theta$ and $h_\phi = r$.

2. (a) From notes, we have $\mathbf{r}(r, \phi, \theta) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)$, it follows that

$$\begin{aligned} \frac{\partial\mathbf{r}}{\partial r} &= (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), \quad h_r = 1 \\ \frac{\partial\mathbf{r}}{\partial\phi} &= (r\cos\phi\cos\theta, r\cos\phi\sin\theta, -r\sin\phi), \quad h_\phi = r, \\ \frac{\partial\mathbf{r}}{\partial\theta} &= (-r\sin\phi\sin\theta, r\sin\phi\cos\theta, 0), \quad h_\theta = r\sin\phi. \end{aligned}$$

Thus the local basis vectors are

$$\begin{aligned}
\hat{\mathbf{r}} &= (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), \\
\hat{\boldsymbol{\phi}} &= (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi), \\
\hat{\boldsymbol{\theta}} &= (-\sin\theta, \cos\theta, 0).
\end{aligned}$$

(b) From the basis vectors in spherical coordinates, one finds that

$$\begin{gathered}
\frac{\partial \hat{\mathbf{r}}}{\partial \phi} = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi) = \hat{\boldsymbol{\phi}}, \quad \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \phi} = (-\sin\phi\cos\theta, -\sin\phi\sin\theta, -\cos\phi) = -\hat{\mathbf{r}} \\
\frac{\partial \hat{\mathbf{r}}}{\partial \theta} = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0) = \sin\phi\ \hat{\boldsymbol{\theta}}, \quad \frac{\partial \hat{\boldsymbol{\phi}}}{\partial \theta} = (-\cos\phi\sin\theta, \cos\phi\sin\theta, 0) = \cos\phi\ \hat{\boldsymbol{\theta}}, \\
\frac{\partial \hat{\boldsymbol{\theta}}}{\partial \theta} = (-\cos\theta, -\sin\theta, 0) = -\sin\phi\ \hat{\mathbf{r}} - \cos\phi\ \hat{\boldsymbol{\phi}}.
\end{gathered}$$

(c) Can do this two ways. First, we can remember the formula derived in the notes, and substitute in the scale factors directly to give

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{1}{r^2\sin\phi}\left[\frac{\partial(u_r r^2\sin\phi)}{\partial r} + \frac{\partial(u_\phi r\sin\phi)}{\partial\phi} + \frac{\partial(u_\theta r)}{\partial\theta}\right]$$

Or, if we can't be bothered to remember the formula, we can calculate the divergence directly from the definition of the gradient, which is easy to remember in terms of scale factors. Then you get

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{u} &= \left(\hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\phi}}}{r}\frac{\partial}{\partial\phi} + \frac{\hat{\boldsymbol{\theta}}}{r\sin\phi}\frac{\partial}{\partial\theta}\right)\cdot\left(u_r\hat{\mathbf{r}} + u_\phi\hat{\boldsymbol{\phi}} + u_\theta\hat{\boldsymbol{\theta}}\right) \\
&= \hat{\mathbf{r}}\cdot\left(\frac{\partial u_r}{\partial r}\hat{\mathbf{r}} + u_r\frac{\partial\hat{\mathbf{r}}}{\partial r} + \frac{\partial u_\phi}{\partial r}\hat{\boldsymbol{\phi}} + u_\phi\frac{\partial\hat{\boldsymbol{\phi}}}{\partial r} + \frac{\partial u_\theta}{\partial r}\hat{\boldsymbol{\theta}} + u_\theta\frac{\partial\hat{\boldsymbol{\theta}}}{\partial r}\right) \\
&+ \frac{\hat{\boldsymbol{\phi}}}{r}\cdot\left(u_r\frac{\partial\hat{\mathbf{r}}}{\partial\phi} + \frac{\partial u_r}{\partial\phi}\hat{\mathbf{r}} + u_\phi\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi} + \frac{\partial u_\phi}{\partial\phi}\hat{\boldsymbol{\phi}} + u_\theta\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\phi} + \frac{\partial u_\theta}{\partial\phi}\hat{\boldsymbol{\theta}}\right) \\
&+ \frac{\hat{\boldsymbol{\theta}}}{r\sin\phi}\cdot\left(\frac{\partial u_\theta}{\partial\theta}\hat{\boldsymbol{\theta}} + u_\theta\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} + \frac{\partial u_r}{\partial\theta}\hat{\mathbf{r}} + +u_r\frac{\partial\hat{\mathbf{r}}}{\partial\theta} + \hat{\boldsymbol{\phi}}\frac{\partial u_\phi}{\partial\theta} + u_\phi\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\theta}\right)
\end{aligned}$$

We need to substitute in from part (b). Some of the terms are zero (e.g. $\partial\hat{\mathbf{r}}/\partial r = 0$ from the definitions of the basis vectors) and others are zero because of orthogonality of the basis vectors. So

$$\begin{aligned}
\boldsymbol{\nabla}\cdot\mathbf{u} &= \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{u_r}{r} + \frac{1}{r\sin\phi}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} + \frac{\cos\phi u_\phi}{r\sin\phi} \\
&= \frac{1}{r^2}\frac{\partial(r^2u_r)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial(\sin\phi u_\phi)}{\partial\phi} + \frac{1}{r\sin\phi}\frac{\partial u_\theta}{\partial\theta}.
\end{aligned}$$

and we have used all of the relations established in part (b).

The two answers are the same and are expanded out as

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{2u_r}{r} + \frac{1}{r}\frac{\partial u_\phi}{\partial\phi} + \frac{\cot\phi u_\phi}{r} + \frac{1}{r\sin\phi}\frac{\partial u_\theta}{\partial\theta}$$

(d) Use formula derived in the notes. So

$$\boldsymbol{\nabla}\times\mathbf{u} = \frac{\hat{\mathbf{r}}}{r^2\sin\phi}\left(\frac{\partial(u_\theta r\sin\phi)}{\partial\phi} - \frac{\partial(u_\phi r)}{\partial\theta}\right) + \frac{\hat{\boldsymbol{\phi}}}{r\sin\phi}\left(\frac{\partial u_r}{\partial\theta} - \frac{\partial(u_\theta r\sin\phi)}{\partial r}\right) + \frac{\hat{\boldsymbol{\theta}}}{r}\left(\frac{\partial(u_\phi r)}{\partial r} - \frac{\partial u_r}{\partial\phi}\right)$$

Or you could calculate it directly using the definition of the gradient as a vector and $\mathbf{u}$. It's very messy.

(e) The Laplacian is $\Delta f = \boldsymbol{\nabla} \cdot \mathbf{u}$ where $(u_r, u_\phi, u_\theta) = \mathbf{u} = \boldsymbol{\nabla} f = (f_r, f_\phi/r, f_\theta/(r\sin\phi))$ according to the lecture notes. So using the (unexpanded) definition of the divergence from part (c) we have

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial\theta^2}$$

after simplifying where we can.

3. (a) We know from lectures $\partial r/\partial x_i = x_i/r$... so

$$\begin{aligned}\Delta\phi = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\phi = \frac{\partial}{\partial x_i}\left(\frac{x_i}{r}f'(r)\right) &= \frac{x_i^2}{r^2}f''(r) + \frac{f'(r)}{r}\frac{\partial x_i}{\partial x_i} - \frac{f'(r)x_i^2}{r^3} \\ &= f''(r) + \frac{3f'(r)}{r} - \frac{f'(r)}{r} = f''(r) + \frac{2f'(r)}{r}.\end{aligned}$$

using $\partial x_i/\partial x_i = 3$, $x_i^2 = r^2$ and so on. This is the same as Q2(e) when $f = f(r)$ – i.e. $f$ independent of $\theta$ and $\phi$. Which is a relief.

(b) There are two ways of doing this. The first is indirect. We start with

$$(\boldsymbol{\mu}\cdot\boldsymbol{\nabla})\left(\frac{1}{r}\right) = \mu_i\frac{\partial}{\partial x_i}\left(\frac{1}{r}\right) = -\mu_i\frac{x_i}{r^3} = -\frac{\boldsymbol{\mu}\cdot\mathbf{r}}{r^3}.$$

Now from Problem Sheet 2, Q4(b), we know $\Delta(r^{-1}) = 0$. Also, $\Delta(\boldsymbol{\mu}\cdot\mathbf{v}) = \boldsymbol{\mu}\cdot\Delta\mathbf{v}$ since $\boldsymbol{\mu}$ is constant. Thus

$$\Delta\left(\frac{\boldsymbol{\mu}\cdot\mathbf{r}}{r^3}\right) = -\Delta\boldsymbol{\mu}\cdot\boldsymbol{\nabla}\left(\frac{1}{r}\right) = -\boldsymbol{\mu}\cdot\boldsymbol{\nabla}(\Delta(r^{-1})) = 0.$$

The second way is a more obvious approach.

$$\begin{aligned}\Delta\frac{\boldsymbol{\mu}\cdot\mathbf{r}}{r^3} &= \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}\left(\frac{\mu_j x_j}{r^3}\right) = \frac{\partial}{\partial x_i}\left(\frac{\mu_j}{r^3}\frac{\partial x_j}{\partial x_i} - \frac{3\mu_j x_j x_i}{r^5}\right) = \frac{\partial}{\partial x_i}\left(\frac{\mu_i}{r^3} - \frac{\mu_j x_j x_i}{r^5}\right) \\ &= -3\frac{\mu_i x_i}{r^5} - 3\frac{\mu_j\delta_{ij}x_i}{r^5} - 3\frac{\mu_j x_j\delta_{ii}}{r^5} + 15\frac{\mu_j x_j x_i x_i}{r^7} \\ &= -3\frac{\boldsymbol{\mu}\cdot\mathbf{r}}{r^5} - 3\frac{\boldsymbol{\mu}\cdot\mathbf{r}}{r^5} - 9\frac{\boldsymbol{\mu}\cdot\mathbf{r}}{r^5} + 15\frac{(\boldsymbol{\mu}\cdot\mathbf{r})r^2}{r^7} = 0.\end{aligned}$$

4. (a) If $\mu$ is constant then we can combine the two equations as

$$\left(\frac{x}{a\cosh\mu}\right)^2 + \left(\frac{y}{a\sinh\mu}\right)^2 = \sin^2\nu + \cos^2\nu = 1$$

and this is an ellipse centred on the origin with semi-major/minor axes $a\cosh\mu$ and $a\sinh\mu$.

Similarly, if $\nu$ is constant, we write

$$\left(\frac{x}{a\cos\nu}\right)^2 - \left(\frac{y}{a\sin\nu}\right)^2 = \cosh^2\mu - \sinh^2\mu = 1$$

which are equations of hyperbolae centred on the origin.

(b) So we have $(x, y) = \mathbf{r}(\mu, \nu) = (a\cosh\mu\cos\nu, a\sinh\mu\sin\nu)$ and so

$$\hat{\boldsymbol{\mu}} = \frac{1}{h_\mu}\frac{\partial\mathbf{r}}{\partial\mu}, \qquad h_\mu = \left\|\frac{\partial\mathbf{r}}{\partial\mu}\right\|, \quad \text{and} \quad \hat{\boldsymbol{\nu}} = \frac{1}{h_\nu}\frac{\partial\mathbf{r}}{\partial\nu}, \qquad h_\nu = \left\|\frac{\partial\mathbf{r}}{\partial\nu}\right\|$$

which means

$$\hat{\boldsymbol{\mu}} = \frac{1}{h_\mu}(a\sinh\mu\cos\nu, a\cosh\mu\sin\nu),$$

$$h_\mu = a\sqrt{\sinh^2\mu\cos^2\nu + \cosh^2\mu\sin^2\nu} = a\sqrt{\sinh^2\mu\cos^2\nu + (1+\sinh^2\mu)\sin^2\nu}$$

which gives the answer. Similarly,

$$\hat{\boldsymbol{\nu}} = \frac{1}{h_\nu}(-a\cosh\mu\sin\nu, a\sinh\mu\cos\nu)$$

$$h_\nu = a\sqrt{\cosh^2\mu\sin^2\nu + \sinh^2\mu\cos^2\nu} = a\sqrt{(1+\sinh^2\mu)\sin^2\nu + \sinh^2\mu\cos^2\nu}.$$

Finally,

$$\hat{\boldsymbol{\mu}}\cdot\hat{\boldsymbol{\nu}} = a^2(-\cosh\mu\sinh\mu\cos\nu\sin\nu + \cosh\mu\sinh\mu\cos\nu\sin\nu) = 0$$

so they are orthogonal.

(c) The Jacobian determinant is

$$J(\mathbf{r}) = \frac{\partial(x,y)}{\partial(\mu,\nu)} = \begin{vmatrix} a\sinh\mu\cos\nu & -a\cosh\mu\sin\nu \\ a\cosh\mu\sin\nu & a\sinh\mu\cos\nu \end{vmatrix} = h_\mu h_\nu = a^2(\sinh^2\mu + \sin^2\nu)$$

(for an orthogonal system, the Jacobian determinant is always the product of the scale factors). The map is inviertible if and only if $J(\mathbf{r}) \neq 0$. It is zero when $\mu = 0$ and $\nu = 0, \pi$. So there are two points in the domain at $(x,y) = (\pm a, 0)$ where the map is singular.

(d) Following notes, we have

$$\boldsymbol{\nabla} f = \frac{1}{h_\mu}\frac{\partial f}{\partial\mu}\hat{\boldsymbol{\mu}} + \frac{1}{h_\nu}\frac{\partial f}{\partial\nu}\hat{\boldsymbol{\nu}} = \frac{1}{a\sqrt{\sinh^2\mu+\sin^2\nu}}\left(\frac{\partial f}{\partial\mu}\hat{\boldsymbol{\mu}} + \frac{\partial f}{\partial\nu}\hat{\boldsymbol{\nu}}\right)$$

(e) According to the formula derived in class for the divergence of a vector $\mathbf{u} = u_\mu\hat{\boldsymbol{\mu}} + u_\nu\hat{\boldsymbol{\nu}}$

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \frac{1}{h_\mu h_\nu}\left[\frac{\partial(u_\mu h_\nu)}{\partial\mu} + \frac{\partial(u_\nu h_\mu)}{\partial\nu}\right]$$

and with $(u_\mu, u_\nu) = \boldsymbol{\nabla} f$ from part (d) we have

$$\Delta f = \frac{1}{a^2(\sinh^2\mu + \sin^2\nu)}\left(\frac{\partial^2 f}{\partial\mu^2} + \frac{\partial^2 f}{\partial\nu^2}\right)$$

That's not so bad. In fact, it's arguably tidier even than cylindrical polar coordinates.

5. (a) We go like this:

$$\Delta(fg) = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}(fg) = \boldsymbol{\nabla}\cdot(f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f) = f\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} g + \boldsymbol{\nabla} f\cdot\boldsymbol{\nabla} g + \boldsymbol{\nabla} g\cdot\boldsymbol{\nabla} f + g\boldsymbol{\nabla}\cdot\boldsymbol{\nabla} f$$

and then we're done.

(b) We have that $\boldsymbol{\nabla} r^2 = 2r\boldsymbol{\nabla} r = 2r\mathbf{r}/r = 2\mathbf{r}$. Then $\Delta r^2 = \boldsymbol{\nabla}\cdot(2\mathbf{r}) = 4$ since $\mathbf{r} = (x,y,0)$. Also $\boldsymbol{\nabla}\log(r) = (1/r)\boldsymbol{\nabla} r = \mathbf{r}/r^2$. So

$$\Delta(\log r) = \boldsymbol{\nabla}\cdot(\mathbf{r}/r^2) = (1/r^2)\boldsymbol{\nabla}\cdot\mathbf{r} + \mathbf{r}\cdot\boldsymbol{\nabla}(1/r^2) = 2/r^2 - (2/r^3)\mathbf{r}\cdot\boldsymbol{\nabla} r = (2/r^2) - (2\mathbf{r}\cdot\mathbf{r})/r^4$$

which is zero since $\mathbf{r}\cdot\mathbf{r} = r^2$. Using part (a) we have

$$\Delta(r^2\log r) = 4\log r + 4\frac{\mathbf{r}\cdot\mathbf{r}}{r^2} + 0 = 4 + 4\log r$$

(c) $\Delta^2(r^2\log r) = \Delta(4 + 4\log r) = 0$ since we've already shown $\Delta(\log r) = 0$.