MATH20901 Multivariable Calculus: Solutions 4 [1]

1. We have that

$$L = \int_C |d\mathbf{r}| = \int_0^{2\pi} |\mathbf{p}'(t)| dt$$

under parametrisation and $\mathbf{p}'(t) = a(1-\cos t, \sin t, 0)$ so

$$L = a\int_0^{2\pi} \sqrt{2-2\cos t}\, dt = a\int_0^{2\pi} \sqrt{4\sin^2(t/2)}\, dt = 4a\left[-\cos(t/2)\right]_0^{2\pi} = 8a$$

So the nail travels exactly 4 diameters of the wheel. If the wheel were not moving along the ground, i.e. only rotating, the nail would travel π diameters (the circumference of the wheel). So it actually doesn't go much further on account of its translation.

2. The curve C is a helix with an axis coinciding with the z-axis. We have that

$$\int_C \mathbf{v}\cdot\, d\mathbf{r} = \int_0^1 \mathbf{v}(\mathbf{p}(t))\cdot \mathbf{p}'(t)\, dt.$$

where $\mathbf{p}(t)$ is the path along the curve and

$$\mathbf{p}'(t) = \left(\frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right), -\frac{\pi}{2}\sin\left(\frac{\pi}{2}t\right), 1\right),$$

whilst

$$\mathbf{v}(\mathbf{p}(t)) = \left(\sin\left(\frac{\pi}{2}t\right), \sin\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}t\right), \sin\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}t\right)t\right).$$

Substituting, we get

$$\int_C \mathbf{v}\cdot d\mathbf{r} = \int_0^1 \left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}t\right) - \frac{\pi}{2}\sin^2\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}t\right)t\right)\, dt.$$

The first two terms in the integrand above may be integrated easily (longhand by substitution $u = \sin(\pi t/2)$ if you need to) to obtain

$$\left[\frac{1}{2}\sin^2\left(\frac{\pi}{2}t\right) - \frac{1}{3}\sin^3\left(\frac{\pi}{2}t\right)\right]_0^1 = \frac{1}{6}.$$

The last term needs integration by parts:

$$\int_0^1 \sin\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}t\right)t\, dt = \frac{1}{2}\int_0^1 \sin(\pi t)t\, dt = \left[-\frac{t}{2\pi}\cos(\pi t)\right]_0^1 + \frac{1}{2\pi}\int_0^1 \cos(\pi t)\, dt = \frac{1}{2\pi}.$$

Therefore,

$$\int_C \mathbf{v}\cdot d\mathbf{r} = \frac{1}{6} + \frac{1}{2\pi}.$$

3. Here $dS = |\hat{\mathbf{n}}dS|$ and we have $(x,y) = \mathbf{s}(r,\theta) = (ra\cos\theta, rb\sin\theta)$ and $D = \{(r,\theta)\,|\,0 < r < 1, 0 < \theta < 2\pi\}$ which means

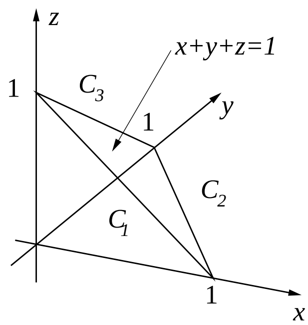
$$\hat{\mathbf{n}}dS = \frac{\partial \mathbf{s}}{\partial r} \times \frac{\partial \mathbf{s}}{\partial \theta} dr d\theta = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a\cos\theta & b\sin\theta & 0 \\ -ra\sin\theta & rb\cos\theta & 0 \end{vmatrix} dr d\theta = rab\, dr d\theta \hat{\mathbf{z}}$$

So the area of the ellipse is $\int_S dS = \int_D |rab\hat{\mathbf{z}}|\, dr d\theta = 2\pi ab \int_0^1 r dr = \pi ab$ In this map of a 2D surface to a 2D surface, we not that the factor rab is just the Jacobian determinant.

4. There are 4 segments to the square: (i) on the path from $(0,0)$ to $(l,0)$, $y = 0$ and $d\mathbf{r} = dx\hat{\mathbf{x}}$; (ii) on the path from $(l,0)$ to (l,l) $x = l$ and $d\mathbf{r} = dy\hat{\mathbf{y}}$; (iii) on the path from (l,l) to $(0,l)$, $y = l$ and $d\mathbf{r} = dx\hat{\mathbf{x}}$; (iv) on the path from $(0,l)$ to $(0,0)$, $x = 0$ and $d\mathbf{r} = dy\hat{\mathbf{y}}$. So we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^l (0,0,0)\cdot\hat{\mathbf{x}}dx + \int_0^l (-l^2 y, ly^2, 0)\cdot\hat{\mathbf{y}}dy + \int_l^0 (-x^2 l, xl^2, 0)\cdot\hat{\mathbf{x}}dx + \int_l^0 (0,0,0)\cdot\hat{\mathbf{y}}dy$$
$$= 0 + l\int_0^l y^2 dy + l\int_0^l x^2 dx + 0 = \frac{2l^4}{3}$$

5. (a) See figure. The plane $x + y + z = 1$ intersects with the plane $y = 0$ along the straightline segment $C_1 = \{y = 0, z = 1 - x\}$, with the plane $z = 0$ along $C_2 = \{z = 0,\ y = 1 - x\}$ and with the plane $x = 0$ along $C_3 = \{x = 0, z = 1 - y\}$.



(b) Need to parametrise the curve C. Do each line segement individually and make sure each segment is oriented in the same sense. So, C_1, C_2, C_3 are described (respectively) by the three paths

$$\mathbf{p}_1(t) = (t, 0, 1-t), \quad \mathbf{p}_2(t) = (1-t, t, 0), \quad \mathbf{p}_3(t) = (0, 1-t, t)$$

each holding for $0 < t < 1$. So

$$\mathbf{p}_1'(t) = (1,0,-1), \quad \mathbf{p}_2'(t) = (-1,1,0), \quad \mathbf{p}_3'(t) = (0,-1,1)$$

First,

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (t^2(1-t), 0, (1-t)^2)\cdot(1,0,-1)\,dt = \int_0^1 (-t^3 - 1 + 2t)\,dt = -\frac{1}{4}$$

Next,

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (0, t^2(1-t), 0)\cdot(-1,1,0)\,dt = \int_0^1 (t^2 - t^3)\,dt = \tfrac{1}{3} - \tfrac{1}{4} = \frac{1}{12}$$

Finally,

$$\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (0,0,t^2)\cdot(0,-1,1)\,dt = \int_0^1 t^2\,dt = \frac{1}{3}$$

Since the curves are all oriented clockwise, we sum over each contribution to give

$$\int_C \mathbf{F}\cdot d\mathbf{r} = -\frac{1}{4} + \frac{1}{12} + \frac{1}{3} = \frac{1}{6}$$

(c) Now the surface integral. Any surface with edges coinciding with the closed curve C will do. Make sense to use the plane $x+y+z=1$. We want to parametrise the curve so we use

$$D = \{(u,v)\,|\,0<v<1-u,\ 0<u<1\}$$

and write $(x,y,z) = \mathbf{s}(u,v) = (u,v,1-u-v)$ (this is just the projection of the slanted triangular section onto the x,y-plane). So

$$\mathbf{N} = \frac{\partial \mathbf{s}}{\partial u}\times\frac{\partial \mathbf{s}}{\partial v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$$

Next,

$$\boldsymbol{\nabla}\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ x^2 z & xy^2 & z^2 \end{vmatrix} = x^2\hat{\mathbf{y}} + y^2\hat{\mathbf{z}}$$

Then

$$\begin{aligned}\int_S \boldsymbol{\nabla}\times\mathbf{F}\cdot dS &= \int_D (u^2\hat{\mathbf{y}} + v^2\hat{\mathbf{z}})\cdot(1,1,1)\,dudv = \int_0^1\int_0^{1-u} u^2+v^2\,dv\,du \\ &= \int_0^1 \left[u^2 v + \frac{v^3}{3}\right]_0^{1-u} du = u^2(1-u) + \frac{(1-u)^3}{3}\,du = \left[\frac{u^3}{3} - \frac{u^4}{4} - \frac{(1-u)^4}{12}\right]_0^1 \\ &= \frac{1}{3} - \frac{1}{4} + \frac{1}{12} = \frac{1}{6}\end{aligned}$$

The same as part (b) by Stokes' theorem.

(d) If $\mathbf{F} = (yz, xz, xy)$ then we see that $\mathbf{F} = \boldsymbol{\nabla}(xyz)$ and hence $\boldsymbol{\nabla}\times\boldsymbol{\nabla}(xyz) = 0$ by an identity. Hence the integral calculated in (c) is zero and (b) is zero also by Stokes' theorem.

6. (a) Since the curve C can be projected onto the unit circle in the (x,y)-plane we parametrised by writing

$$\mathbf{p}(t) = (\cos t, \sin t, 2-\sin t), \qquad 0<t<2\pi$$

Then $\mathbf{F}(\mathbf{p}(t)) = (-\sin^2 t, \cos t, (2-\sin t)^2)$, whilst $\mathbf{p}'(t) = (-\sin t, \cos t, -\cos t)$ and so

$$\begin{aligned}\int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} (-\sin^2 t, \cos t, (2-\sin t)^2)\cdot(-\sin t, \cos t, -\cos t)\,dt \\ &= \int_0^{2\pi} \left(-\sin^3 t + \cos^2 t - \cos t(2-\sin t)^2\right)\,dt \\ &= \int_0^{2\pi} \left(-\sin t(1-\cos^2 t) + \frac{1}{2} + \frac{1}{2}\cos 2t - \cos t(2-\sin t)^2\right)\,dt \\ &= \left[\cos t - \frac{\cos^3 t}{3} + \frac{t}{2} + \frac{\sin 2t}{4} + \frac{(2-\sin t)^3}{3}\right]_0^{2\pi} = \pi\end{aligned}$$

(could have spotted that only the $\cos^2 t$ counts to this integral and made the calculation shorter.)

(b) We project the surface onto the unit circle in to (x, y)-plane so define $D = \{(r, \theta) \,|\, 0 < r < 1,\ 0 < \theta < 2\pi\}$ and define the surface S with

$$\mathbf{s}(r, \theta) = (r\cos\theta, r\sin\theta, (2 - r\sin\theta))$$

Then

$$\mathbf{N} = \frac{\partial \mathbf{s}}{\partial r} \times \frac{\partial \mathbf{s}}{\partial \theta} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \cos\theta & \sin\theta & -\sin\theta \\ -r\sin\theta & r\cos\theta & -r\cos\theta \end{vmatrix} = r\hat{\mathbf{y}} + r\hat{\mathbf{z}}$$

Easy to show (follow answer to Q5(c)) that $\boldsymbol{\nabla} \times \mathbf{F} = (1 + 2y)\hat{\mathbf{z}} = (1 + 2r\sin\theta)\hat{\mathbf{z}}$. So

$$\int_S \boldsymbol{\nabla} \times \mathbf{F} \cdot d\mathbf{S} = \int_D r(1 - 2r\sin\theta)\, dr\, d\theta = \int_0^{2\pi} \int_0^1 (r + 2r^2 \sin\theta)\, dr\, d\theta = \pi - \frac{2}{3}\left[\cos\theta\right]_0^{2\pi} = \pi$$

Same as (a) by Stokes' theorem.

7. Given D we have

$$\begin{aligned} \int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx\, dy &= \int_c^d \int_a^b \frac{\partial g}{\partial x}\, dx\, dy - \int_a^b \int_c^d \frac{\partial f}{\partial y}\, dy\, dx \\ &= \int_c^d \left[g(x, y)\right]_{x=a}^{x=b} dy - \int_a^b \left[f(x, y)\right]_{y=c}^{y=d} dx \\ &= \int_c^d g(b, y)\, dy + \int_d^c g(a, y)\, dy + \int_a^b f(x, c)\, dx + \int_b^a f(x, d)\, dx \end{aligned}$$

after reversing the limits to absorb minus signs. We see that the four integrals circumnavigate the edge of the rectangle in an anticlockwise sense.

In other words

$$\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right)(x, y)\, dx\, dy = \int_{\partial D} (f, g) \cdot d\mathbf{r} = \int_{\partial D} f\, dx + g\, dy.$$

8. Two ways of doing this: (i) We note that $f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f = \boldsymbol{\nabla}(fg)$ and since $\boldsymbol{\nabla} \times \boldsymbol{\nabla}(fg) = 0$ regardless of f, g then

$$\int_C \boldsymbol{\nabla}(fg) \cdot d\mathbf{r} = \int_S \boldsymbol{\nabla} \times \boldsymbol{\nabla}(fg) \cdot d\mathbf{S} = 0$$

(ii) We parametrise C by $\mathbf{p}(t)$, $t_1 < t < t_2$ and $\mathbf{p}(t_1) = \mathbf{p}(t_2)$ since C is closed and so

$$\int_C \boldsymbol{\nabla}(fg) \cdot d\mathbf{r} = \int_{t_1}^{t_2} \boldsymbol{\nabla}(fg)(\mathbf{p}(t)) \cdot \mathbf{p}'(t)\, dt = \int_{t_1}^{t_2} \frac{d(fg)}{dt}(\mathbf{p}(t))\, dt = (fg)(\mathbf{p}(t_2)) - (fg)(\mathbf{p}(t_1)) = 0$$

which is the fundamental theorem of calculus, as in the notes.