MATH20901 Multivariable Calculus: Solutions 5 [1]

1. Stokes' theorem is $\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S} = \int_C \mathbf{v} \cdot d\mathbf{r}$.

   (i) The surface integral over the hemisphere. This is best described using a polar coordinate parametrisation (but there are other ways), so we let $D = \{(r, \theta) \,|\, 0 < r < 3,\ 0 < \theta < 2\pi\}$ and describe the surface S using

$$(x, y, z) = \mathbf{s}(r, \theta) = (r\cos\theta, r\sin\theta, \sqrt{3^2 - r^2})$$

   Then we have

$$\frac{\partial \mathbf{s}}{\partial r} = \left(\cos\theta, \sin\theta, -\frac{r}{(9-r^2)^{1/2}}\right), \quad \frac{\partial \mathbf{s}}{\partial \theta}(r, \theta) = (-r\sin\theta, r\cos\theta, 0)\,.$$

   and so

$$\mathbf{N}(r, \theta) = \frac{\partial \mathbf{s}}{\partial r} \times \frac{\partial \mathbf{s}}{\partial \theta} = \left(\frac{r^2\cos\theta}{(9-r^2)^{1/2}}, \frac{r^2\sin\theta}{(9-r^2)^{1/2}}, r\right).$$

   The curl of $\mathbf{v}$ is given by $\nabla \times \mathbf{v} = (0, 0, -2)$. So $(\nabla \times \mathbf{v}) \cdot \mathbf{N}(r, \theta) = -2r$ and

$$\int_S \nabla \times \mathbf{v} \cdot d\mathbf{S} = \int_D -2r\, dr d\theta = -2\int_0^{2\pi}\int_0^3 r dr d\theta = -18\pi.$$

   (ii) The integral around the boundary. Here $\mathbf{p}(\theta) = \mathbf{s}(3, \theta) = (3\cos\theta, 3\sin\theta, 0)$, $0 < \theta < 2\pi$. Then $\mathbf{p}'(\theta) = (-3\sin\theta, 3\cos\theta, 0)$ whilst $\mathbf{v}(\mathbf{p}(\theta)) = (3\sin\theta, 3\cos\theta, 0)$ so

$$\int_{\partial S} \mathbf{v} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{v}(\mathbf{p}(\theta)) \cdot \frac{d\mathbf{p}}{d\theta}\, d\theta = \int_0^{2\pi} (-9\sin^2\theta - 9\cos^2\theta)\, d\theta = -18\pi.$$

2. From the defintion, $\mathbf{F} = (f_x, f_y, 0)$ and $d\mathbf{r} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$ so

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C f_x dx + f_y dy = \int_D (-f_{xy} + f_{xy}) dx dy = 0$$

   where D is the area enclosed by C and we have simply used Green's theorem in the plane. Of course, this result is proved via other means in the lectures.

3. The integral is $\int_C (-x^2y, xy^2, 0) \cdot d\mathbf{r}$ where C is a square of length l with one vertex on $(0, 0)$. According to Solution Sheet 4, Q4, the value of the integral is $2l^3/3$. We can calculate this integral using Green's theorem in the plane so that the value is

$$\int_D -\frac{\partial(-x^2y)}{\partial y} + \frac{\partial(xy^2)}{\partial x} dx dy = \int_D (x^2 + y^2) dx dy$$

Firstly we note that we can calculate this for $D = \{(0, l) \times (0, l)\}$ since it is

$$\int_0^l dy \int_0^l x^2 dx + \int_0^l dx \int_0^l y^2 dy = 2l^3/3$$

But, to answer the question, we see that the integrand is $x^2 + y^2$ depends only on the distance from $(0, 0)$ and not the angle. So the answer will be the same for any square with its vertex on the origin.

4. (a) The conical surface is best parametrised by polar coordinates so we let $D = \{(r, \theta) \,|0 < r < 1\ , 0 < \theta < 2\pi\}$ and

$$\mathbf{s} = (r \cos \theta, r \sin \theta, r)$$

since $z = \sqrt{x^2 + y^2} = r$ describes the cone. Now

$$\frac{\partial \mathbf{s}}{\partial r} = (\cos \theta, \sin \theta, 1)\,, \quad \frac{\partial \mathbf{s}}{\partial \theta}(r, \theta) = (-r \sin \theta, r \cos \theta, 0)\,.$$

and so

$$\mathbf{N}(r, \theta) = \frac{\partial \mathbf{s}}{\partial r} \times \frac{\partial \mathbf{s}}{\partial \theta} = (-r \cos \theta, -r \sin \theta, r)$$

BUT wait ! We see that $\mathbf{N}$ points inwards towards the axis of the cone, and so we reverse the sign of $\mathbf{N}$ to ensure it points outwards as directed (this is all about the ambiguity of normals to surfaces and in which order cross products are done in the definition of $\mathbf{N}$). So,

$$\mathbf{N}(r, \theta) = (r \cos \theta, r \sin \theta, -r)$$

Now we have, under the parametrisation

$$\mathbf{F}(\mathbf{s}(r, \theta)) = (r \cos \theta, r \sin \theta, r^4)$$

and so

$$\begin{aligned} \int_S \mathbf{F} \cdot d\mathbf{S} &= \int_D \mathbf{F}(\mathbf{s}(r, \theta)) \cdot \mathbf{N}(r, \theta)\, dr d\theta = \int_0^{2\pi} \int_0^1 (r \cos \theta, r \sin \theta, r^4) \cdot (\cos \theta, \sin \theta, -1)\, r\, dr d\theta \\ &= \int_0^{2\pi} \int_0^1 (r^2 - r^5)\, dr d\theta = 2\pi \left[\frac{r^3}{3} - \frac{r^6}{6} \right]_0^1 = \frac{\pi}{3} \end{aligned}$$

(b) Here $\boldsymbol{\nabla} \cdot \mathbf{F} = 1 + 1 + 4z^3 = 2 + 4z^3$. So we have making the standard transformation to cylindrical coordinates (as in Calculus 1)

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F}\, dx dy dz = \int_0^{2\pi} \int_0^1 \int_0^z (2 + 4z^3)\, r dr dz d\theta = 2\pi \int_0^1 (2 + 4z^3) \left[\tfrac{1}{2} r^2\right]_0^z dz = 4\pi/3.$$

(c) By the divergence theorem, we have that the volume integral must equal the integral of the enclosing surface (with normal outwards). It follows then that

$$\int_{z=1, x^2+y^2<1} \mathbf{F} \cdot d\mathbf{S} = \int_V \boldsymbol{\nabla} \cdot \mathbf{F} dV - \int_S \mathbf{F} \cdot d\mathbf{S} = 4\pi/3 - \pi/3 = \pi.$$

Check this. So the 'lid' of the cone is parametrised by $D = \{(r, \theta)\,|\,0 < r < 1,\ 0 < \theta < 2\pi\}$, $\mathbf{s}(r, \theta) = (r \cos \theta, r \sin \theta, 1)$ and this gives $\mathbf{N} = r\hat{\mathbf{z}}$ so that

$$\int_{z=1, x^2+y^2<1} \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 (r \cos \theta, r \sin \theta, 1) \cdot (r\hat{\mathbf{z}})\, dr d\theta = \int_0^{2\pi} \int_0^1 r dr d\theta = \pi$$

5. We have $\mathbf{F} = (x, y, -z)$ so $\boldsymbol{\nabla} \cdot \mathbf{F} = 1 + 1 - 1 = 1$. So by the divergence theorem (the easiest of the two approaches)

$$\int_{\partial V} \mathbf{F} \cdot d\mathbf{S} = \int_V \boldsymbol{\nabla} \cdot \mathbf{F} dV = \int_0^1 \int_0^1 \int_0^1 dx dy dz = 1$$

If we want to calculate the LHS of the above directly, we need to divide the surface into 6 faces of the cube. So let S_1 be the the face in the plane $x = 1$. On S_1, $d\mathbf{S} = \hat{\mathbf{x}} dy dz$ and

$$\int_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 (1, y, -z) \cdot \hat{\mathbf{x}} \, dy dz = 1.$$

Next S_{-1} the parallel face in the plane $x = 0$. On S_{-1}, $\mathbf{S} = -\hat{\mathbf{x}} dy dz$ since the normal is outwards from the cube. Here

$$\int_{S_{-1}} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 (0, y, -z) \cdot (-\hat{\mathbf{x}}) \, dy dz = 0$$

We continue like this defining S_2 and S_{-2} to be the faces in the planes $y = 1$ and $y = 0$, S_3 and S_{-3} the faces in the planes $z = 1$ and $z = 0$ and we find

$$\int_{S_2} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 (x, 1, -z) \cdot \hat{\mathbf{y}} \, dx dz = 1, \quad \int_{S_{-2}} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 (x, 0, -z) \cdot (-\hat{\mathbf{y}}) \, dx dz = 0$$

and

$$\int_{S_3} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 (x, y, -1) \cdot \hat{\mathbf{z}} \, dx dz = -1, \quad \int_{S_{-3}} \mathbf{F} \cdot d\mathbf{S} = \int_0^1 \int_0^1 (x, y, 0) \cdot (-\hat{\mathbf{z}}) \, dx dz = 0$$

The sum of all 6 faces gives

$$\int_{\partial V} \mathbf{F} \cdot d\mathbf{S} = 1 + 0 + 1 + 0 - 1 + 0 = 1.$$

6. Probably worth doing again from scratch. So $1/r = 1/\sqrt{x^2 + y^2 + z^2} = 1/|\mathbf{r}|$ and (this is long hand)

$$\boldsymbol{\nabla} \left(\frac{1}{r} \right) = \left(-\frac{x}{r^3} - \frac{y}{r^3}, -\frac{z}{r^3} \right) = \frac{-\mathbf{r}}{r^3} \equiv \frac{-\hat{\mathbf{r}}}{r^2}$$

since $\hat{\mathbf{r}} = \mathbf{r}/r$. It follows that

$$\Delta \left(\frac{1}{r} \right) = \boldsymbol{\nabla} \cdot \left(\frac{-\mathbf{r}}{r^3} \right) = \frac{(3x^2 - r + 3y^2 - r + 3z^2 - r)}{r^5} = 0.$$

Or we can use suffices, or the Laplacian in spherical polars is even better. Now

$$\int_V \Delta \left(\frac{1}{r} \right) dV = \int_V \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \left(\frac{1}{r} \right) dV = \int_S \boldsymbol{\nabla} \left(\frac{1}{r} \right) \cdot d\mathbf{S}$$

If the volume V is a sphere of radius R then $d\mathbf{S} = \hat{\mathbf{r}} R^2 \sin\phi d\theta d\phi$, so

$$\int_V \Delta \left(\frac{1}{r} \right) dV = -\int_0^{2\pi} \int_0^{\phi} \frac{\hat{\mathbf{r}}}{R^2} \cdot \hat{\mathbf{r}} R^2 \sin\phi \, d\phi d\theta = -2\pi \left[-\cos\phi \right]_0^{\pi} = -4\pi$$

since $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = 1$.

The integrand on the LHS is zero for all $r \neq 0$ and yet the integral over the volume of an arbitrary sized sphere is -4π. Have we made a mistake ? No, it turns out that the singularity at $r = 0$ is enough to make this happen. In physics we say that there is a *point source* at the origin and it turns out that we can extend the definition of $\Delta(1/r)$ to $r = 0$ by writing

$$\Delta(1/r) = -4\pi\delta(\mathbf{r})$$

where δ is a Dirac-delta function. You see this in APDE2 and Fluids for e.g.

7. (a) Use the divergence theorem to write

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{E}\, dV = 4\pi \int_V \rho(\mathbf{r})\, dV$$

Then, obviously,

$$\int_V (\boldsymbol{\nabla} \cdot \mathbf{E} - 4\pi\rho(\mathbf{r}))\ dV = 0$$

and since this is true for any volume V is must be that the integrand is zero everywhere and hence

$$\boldsymbol{\nabla} \cdot \mathbf{E} = 4\pi\rho(\mathbf{r}), \qquad \text{for all points } \mathbf{r}$$

(b) $\mathbf{E} = (x, y, z)$ so $\boldsymbol{\nabla} \cdot \mathbf{E} = 1 + 1 + 1 = 3$ and so

$$Q = \int_V 3\, dV = 3.8 = 24$$

as V is the cuboid with three equal sides of length 2.

8. So we use the divergence theorem to write

$$\frac{1}{3}\int_S \mathbf{r} \cdot \hat{\mathbf{n}} dS = \int_S \frac{1}{3}(x, y, z) \cdot d\mathbf{S} = \int_V \boldsymbol{\nabla} \cdot \left(\frac{1}{3}(x, y, z)\right) dV = \int_V (1/3 + 1/3 + 1/3) dV = \int_V dV$$

and that's it.

The volume of a sphere of radius a is $\frac{4}{3}\pi a^3$. Using the LHS, noting that $\hat{\mathbf{n}} \equiv \hat{\mathbf{r}}$ and that $\mathbf{r} = |\mathbf{r}|\hat{\mathbf{r}} = a\hat{\mathbf{r}}$ we have

$$\frac{1}{3}\int_S \mathbf{r} \cdot \hat{\mathbf{n}}\, dS = \frac{a}{3}\int_S \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\, dS = \frac{a}{3}(4\pi a^2) = \tfrac{4}{3}\pi a^3$$

since the integral is the surface area of the sphere.

9. Here we want to use the divergence theorem to compute

$$\int_S dS = \int_S \mathbf{v} \cdot \hat{\mathbf{n}}\, dS = \int_V \boldsymbol{\nabla} \cdot \mathbf{v}\, dV$$

This works if we can find a $\mathbf{v}$ which coincides with $\hat{\mathbf{n}}$ on S. This can't be done generally.

10. We have from the notes, starting from the second term in the equation below and going left and right

$$\int_V (\boldsymbol{\nabla} v \cdot \boldsymbol{\nabla} u + v\Delta u)\, dV = \int_V \boldsymbol{\nabla} \cdot (v\boldsymbol{\nabla} u)\, dV = \int_S (v\boldsymbol{\nabla} u) \cdot \hat{\mathbf{n}}\, dS = 0$$

and if $\Delta u = 0$ in V and $v = 0$ on S then we get the result as stated.