MATH20901 Multivariable Calculus: Problems 1 [1]

1. Which of these are linear maps and why ?

   $$\text{(i) } \mathbf{F}(\mathbf{x}) = (x_3, x_1, x_2), \quad \text{(ii) } \mathbf{F}(\mathbf{x}) = (x_3 x_1, x_1 x_2, x_2 x_3), \quad \text{(iii) } \mathbf{F}(\mathbf{x}) = (x_3 + x_1, x_2 + x_3)$$

   In all cases, $\mathbf{x} = (x_1, x_2, x_3)$. If the map is linear, write down the matrix A which defines the map.

2. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $\mathbf{x} \mapsto (-x_2, x_1)$ and $\mathbf{G}: \mathbb{R}^2 \to \mathbb{R}^2$ s.t $\mathbf{x} \mapsto (x_2, \sin x_1)$. Evaluate $\mathbf{G} \circ \mathbf{F}$ and $\mathbf{F} \circ \mathbf{G}$.

3. Let $\mathbf{F}: \mathbb{R}^2 \to \mathbb{R}^3$ be given by

   $$\mathbf{F}(\mathbf{x}) = \left(x_1^2 x_2, \sin(x_1 + x_2), e^{x_1 x_2}\right)$$

   (a) Compute the matrix $\mathbf{F}'(\mathbf{x})$ of partial derivatives

   $$\frac{\partial F_i}{\partial x_j}.$$

   (b) Compute the directional derivative $D_{\mathbf{v}}\mathbf{F}(\mathbf{x})$, where $\mathbf{v} = (1, 2)$.

   (c) Compare the result from (b), evaluated at $\mathbf{x} = \mathbf{x}_0 = (1, 1)$, to the vector $\mathbf{F}'(\mathbf{x}_0)\mathbf{v}$.

4. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $\mathbf{F}(\mathbf{x}) = A\mathbf{x}$, where

   $$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

   Let $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

   $$\mathbf{G}(\mathbf{x}) = (x_1 x_2, x_2 x_3, \sin(x_1 x_2 x_3)).$$

   Let $\mathbf{H} = \mathbf{G} \circ \mathbf{F}$.

   (a) Use the chain rule to calculate $\mathbf{H}'(1, 1)$.

   (b) Calculate $\mathbf{H}'(1, 1)$ directly.

5. Consider the coupled nonlinear system of equations given by

   $$x^3 + e^y = s, \qquad \cos x + xy = t$$

   which we wish to be able to solve uniquely for (x, y) in terms of (s, t). Show this cannot be done at $(x, y) = (0, 0)$.

6. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be s.t. $\mathbf{F}(\mathbf{x}) = (x^2 + y^2, x + y^3/x)$ where $\mathbf{x} = (x, y)$. Show that close to the point $\mathbf{x} = (1, 2)$,

$$\mathbf{F}(\mathbf{x}) \approx (2x + 4y - 5, -7x + 2y - 8)$$

7. Show that the pair of equations

$$x^2 + y^2 - yu^2 + v^2 - 5 = 0, \qquad xy^2 - yv/u - 3u = 0,$$

determine local functions $u(x, y)$ and $v(x, y)$ defined for (x, y) near $(2, 1)$ such that $u(2, 1) = 1$ and $v(2, 1) = -1$. Compute $\partial v/\partial y$ at $(x, y) = (2, 1)$.

8. The transformation from spherical to Cartesian co-ordinates is defined by the map

$$(x, y, z) = \mathbf{r}(r, \phi, \theta) = (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi).$$

   (a) Calculate the derivative of the map $\mathbf{r}'$.

   (b) Show that the Jacobian determinant is given by

$$J_{\mathbf{r}} \equiv \frac{\partial(x, y, z)}{\partial(r, \phi, \theta)} = r^2 \sin\phi.$$

   When can we solve for (r, ϕ, θ) in terms of (x, y, z)?

9. (Calculus 1 revision) Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(\mathbf{x}) = \begin{cases} \dfrac{xy^3}{x^2 + y^6}, & \mathbf{x} \neq 0, \\ 0, & \mathbf{x} = 0. \end{cases}$$

   (a) Show that f is continuous for $\mathbf{x} \neq 0$.

   (b) Show that f is not continuous at the origin. *Hint: consider the path $x = y^3$.*