MATH20901 Multivariable Calculus: Problems 2 [1]

1. (a) Let δ_{ij} be the n-dimensional delta symbol. Compute $\delta_{ij}\delta_{ij}$, using the summation convention.

   (b) Define three matrices $A \in \mathbb{R}^{n\times p}$, $B \in \mathbb{R}^{q\times p}$, and $C \in \mathbb{R}^{q\times s}$. Express the i, jth component of the product AB^TC in terms of the components of A, B, C using Eisntein summation convention.

2. The vectors $\mathbf{e}_i$, $1 \le i \le m$ form a non-normal orthogonal basis in $\mathbb{R}^m$ (i.e. $|\mathbf{e}_i| \ne 1$). This means that for every $\mathbf{x}$ in $\mathbb{R}^m$ there exist c_j, $1 \le j \le m$ such that

$$\mathbf{x} = c_j\mathbf{e}_j.$$

   Express c_j in terms of $\mathbf{x}$.

   If the vectors $\mathbf{e}_i$, $1 \le i \le m$ form a non-orthogonal basis in $\mathbb{R}^m$, how would you compute the c_j now ?

3. Compute

   (a) the gradient of $f(\mathbf{r}) = \cos(xy) + \cos(yz)$, and verify that $\boldsymbol{\nabla} \times \boldsymbol{\nabla} f = 0$ in this case.

   (b) the divergence of $\mathbf{u}(\mathbf{r}) = (x \sin z, yz, \cos z)$,

   (c) the curl of $\mathbf{v}(\mathbf{r}) = (ayz, bzx, cxy)$, and verify that $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla} \times \mathbf{v} = 0$ in this case.

4. In each case of the following cases, use suffix notation. Compute

   (a) the gradient of $f(\mathbf{r}) = \mathbf{a} \cdot \mathbf{r}$, where $\mathbf{a} \in \mathbb{R}^3$ is a fixed vector.

   (b) the divergence of $\mathbf{v}(\mathbf{r}) = \boldsymbol{\nabla} r^n$ where $r = |\mathbf{r}|$. For which value of n does the divergence vanish ?

   (c) The curl of $\mathbf{v}(\mathbf{r}) = \boldsymbol{\omega} \times \mathbf{r}$, where $\boldsymbol{\omega} \in \mathbb{R}^3$ is a fixed vector.

5. (a) Let $f(r)$ be a smooth scalar-valued function of $r = |\mathbf{r}|$, and let $\mathbf{a} \in \mathbb{R}^3$ be a constant vector. Use suffix notation to calculate

$$\text{(i) } \boldsymbol{\nabla} \times (\mathbf{r} \times \mathbf{a} f(r))\,, \qquad \text{and (ii) } \boldsymbol{\nabla} \cdot (\mathbf{a} f(r))$$

   (b) Let $\mathbf{u}$ be a vector field. Show, using suffix notation, that

$$\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}) = \frac{1}{2}\boldsymbol{\nabla}(\mathbf{u} \cdot \mathbf{u}) - (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}.$$

6. Prove the following identities for scalar functions f, g and the vector function $\mathbf{v}$:

   (a) $\boldsymbol{\nabla} \cdot (f\mathbf{v}) = f\boldsymbol{\nabla} \cdot \mathbf{v} + \boldsymbol{\nabla} f \cdot \mathbf{v}$;

   (b) $\boldsymbol{\nabla} \cdot (f\boldsymbol{\nabla} g) = f\Delta g + \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g$;

(c) $\nabla \times (f\mathbf{v}) = f\nabla \times \mathbf{v} + \nabla f \times \mathbf{v}$.

7. Without doing any calculations, how do you know the following result is false ?

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{u} \cdot \nabla \times \mathbf{v} + \mathbf{v} \cdot \nabla \times \mathbf{u}.$$

Find a corrected version of this result.

8. Show that, for vector fields $\mathbf{u}(\mathbf{r})$, $\mathbf{v}(\mathbf{r})$ both in $\mathbb{R}^3$

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = (\nabla \cdot \mathbf{v})\mathbf{u} - (\nabla \cdot \mathbf{u})\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v}.$$

9. (a) The force per unit volume of gravity on a particle of density ρ is given by

$$\mathbf{F} = -\rho g \hat{\mathbf{z}}$$

where g is gravitational acceleration. Find a ϕ such that $\mathbf{F} = \nabla\phi$.

(b) Euler's equations governing the motion of a so-called 'ideal fluid' are given by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p, \qquad \text{with} \qquad \nabla \cdot \mathbf{u} = 0$$

where p is a scalar (pressure) function, t is time and $\mathbf{u}$ is a vector function (the fluid velocity).

Take the curl of Euler's equation, using the results of Q5(b) and Q8 to derive the following equation for the *vorticity* $\boldsymbol{\omega} = \nabla \times \mathbf{u}$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}.$$

(c) Deduce that if $\mathbf{u}$ is two-dimensional (e.g. no component in the $\hat{\mathbf{z}}$ direction) then $\boldsymbol{\omega}$ is constant.

10. Navier's equation governs the motion of a linear isotropic elastic solid and is given by

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu\Delta\mathbf{u}$$

where ρ is the density of the elastic solid, λ and μ are material constants, t is time and $\mathbf{u}$ is the (small) displacement of the solid as a function of space and time.

Assuming space and time derivatives are interchangeable, apply the divergence and curl operators to Navier's equation to show that $\phi = \nabla \cdot \mathbf{u}$ and $\mathbf{H} = \nabla \times \mathbf{u}$ satisfy

$$\frac{\partial^2 \phi}{\partial t^2} = c_1^2\Delta\phi, \qquad \text{and} \qquad \frac{\partial^2 \mathbf{H}}{\partial t^2} = c_2^2\Delta\mathbf{H},$$

where c_1 and c_2 are to be found.

[Hint: You will need to use results from lectures, namely: (i) $\Delta\mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u})$; (ii) $\nabla \times \nabla f = 0$; (iii) $\nabla \cdot (\nabla \times \mathbf{v}) = 0$ for any f, $\mathbf{v}$.]