

MATH20901 Multivariable Calculus: Problems 3¹

1. Let $(x, y, z) = \mathbf{r}(\mathbf{q}) = \mathbf{r}(q_1, q_2, q_3)$ define an *orthogonal* coordinate system.

- Write down the Jacobian matrix $\mathbf{r}'(\mathbf{q})$ and express the result in terms of the orthogonal basis vectors $\hat{\mathbf{q}}_\alpha$.
- Show that the inverse of the Jacobian matrix is

$$(\mathbf{r}'(\mathbf{q}))^{-1} = \begin{pmatrix} \frac{1}{h_1^2} \frac{\partial x}{\partial q_1} & \frac{1}{h_1^2} \frac{\partial y}{\partial q_1} & \frac{1}{h_1^2} \frac{\partial z}{\partial q_1} \\ \frac{1}{h_2^2} \frac{\partial x}{\partial q_2} & \frac{1}{h_2^2} \frac{\partial y}{\partial q_2} & \frac{1}{h_2^2} \frac{\partial z}{\partial q_2} \\ \frac{1}{h_3^2} \frac{\partial x}{\partial q_3} & \frac{1}{h_3^2} \frac{\partial y}{\partial q_3} & \frac{1}{h_3^2} \frac{\partial z}{\partial q_3} \end{pmatrix}.$$

[Hint: an orthogonal matrix R has inverse R^T – §2.8.1 of notes].

- Use the result from (b) to calculate $\frac{\partial \phi}{\partial y}$, where ϕ is the latitudinal angle in spherical coordinates.
- Spherical coordinates are defined by

$$\mathbf{r}(r, \phi, \theta) = (r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi).$$

- Calculate the basis vectors $\hat{\mathbf{r}}$, $\hat{\phi}$, and $\hat{\theta}$, as well as the scale factors h_r , h_ϕ , and h_θ .
- Show that

$$\frac{\partial \hat{\mathbf{r}}}{\partial \phi} = \hat{\phi}, \quad \frac{\partial \hat{\phi}}{\partial \phi} = -\hat{\mathbf{r}}, \quad \frac{\partial \hat{\mathbf{r}}}{\partial \theta} = \sin \phi \hat{\theta}, \quad \frac{\partial \hat{\phi}}{\partial \theta} = \cos \phi \hat{\theta}, \quad \frac{\partial \hat{\theta}}{\partial \theta} = -\sin \phi \hat{\mathbf{r}} - \cos \phi \hat{\phi}.$$

- Calculate $\nabla \cdot \mathbf{u}$ in spherical coordinates, where $\mathbf{u} = u_r \hat{\mathbf{r}} + u_\phi \hat{\phi} + u_\theta \hat{\theta}$.
- Treat yourself. How about $\nabla \times \mathbf{u}$?
- Finally, write down Δf in spherical polars.
- (a) Let $\phi(\mathbf{r}) = f(r)$, where f is a function of a single variable, $r = |\mathbf{r}|$. Compute $\Delta \phi$.
- (b) Let $\boldsymbol{\mu} \in \mathbb{R}^3$ be a nonzero vector. Compute $\Delta \frac{\boldsymbol{\mu} \cdot \mathbf{r}}{r^3}$.

4. This question concerns the transformation to elliptical coordinates (μ, ν) given by the relation

$$x = a \cosh \mu \cos \nu, \quad y = a \sinh \mu \sin \nu$$

where $\mu \in [0, \infty)$, $\nu \in [0, 2\pi)$.

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(a) Show that curves of constant μ correspond to ellipses in the (x, y) -plane and that curves of constant ν are hyperbolae.

(b) Derive a basis $\hat{\mu}, \hat{\nu}$ for elliptical coordinates and, in doing so, show that the scale factors are

$$h_\mu = h_\nu = a\sqrt{\sinh^2 \mu + \sin^2 \nu}.$$

Also confirm that $\hat{\mu}$ and $\hat{\nu}$ are orthogonal.

(c) Calculate the Jacobian determinant. Is the mapping of coordinates always invertible ? If not, when is it non-invertible ?

(d) Express ∇f in elliptical coordinates.

(e) Find Δf in elliptical coordinates.

5. (a) For the two scalar function f, g , derive the relation $\Delta(fg) = f\Delta g + g\Delta f + 2\nabla f \cdot \nabla g$
(b) Assuming that we are working in 2D so that $\mathbf{r} = (x, y, 0)$ show that $\Delta(r^2 \log r) = 4 + 4 \log r$.
(c) Hence show that $\Delta^2(r^2 \log r) = 0$ where $\Delta^2 \equiv \Delta\Delta$ is called the biharmonic operator.