MATH20901 Multivariable Calculus: Problems 4 [1]

1. The length of a curve C is given by $\int_C |d\mathbf{r}|$. A cycloid is the path traced out in space by, say, a nail in a tyre on a wheel of radius a. The path is given by $\mathbf{p}(t) = a(t - \sin t, 1 - \cos t, 0)$ with $0 < t < 2\pi$ for one revolution. Find the distance the nail travels during one revolution of the wheel.

2. Let $\mathbf{v}(\mathbf{r}) = (x, xy, xyz)$ and let the curve C be parametrised by the path
$$\mathbf{p}(t) = \left(\sin\left(\frac{\pi}{2}t\right), \cos\left(\frac{\pi}{2}t\right), t\right), \quad 0 \le t \le 1.$$
Evaluate the line integral $\int_C \mathbf{v} \cdot d\mathbf{r}$.

3. The surface area of an object with surface S is given by $\int_S dS$. Use a transformation of coordinates $x = ra\cos\theta$, $y = rb\sin\theta$ to show that the area of an ellipse with semi-major and minor axes a and b is πab.

4. Calculate the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ when $\mathbf{F} = (-x^2y, xy^2, 0)$ and C is a square in the $(x, y)-$plane with vertices at $(0, 0)$, $(l, 0)$, (l, l), $(0, l)$ which is oriented anticlockwise.

5. Let $\mathbf{F}(\mathbf{r}) = (x^2z, xy^2, z^2)$ and define a closed curve C in $\mathbb{R}^3$ comprised of three straight line segments C_1, C_2 and C_3 formed by the intersection of the plane $x + y + z = 1$ with the planes $y = 0$, $z = 0$ and $x = 0$ (respectively).

   (a) First you should sketch the curve $C = C_1 \cup C_2 \cup C_3$ in the (x, y, z)-space.

   (b) Calculate $\int_C \mathbf{F} \cdot d\mathbf{r}$, by considering each line segment separately. *[Hint: be careful to make sure the orientation of each segment of C is aligned with the other segments.]*

   (c) Calculate $\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}$ over an appropriately-defined surface with boundary $\partial S = C$.

   (d) If, instead, $\mathbf{F} = (yz, xz, xy)$ explain how you could have calculated parts (b) and (c) instantly.

6. Let $\mathbf{F}(\mathbf{r}) = (-y^2, x, z^2)$ and define a curve C in $\mathbb{R}^3$ to be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.

   (a) Compute
$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

(b) Compute

$$\int_S \boldsymbol{\nabla} \times \mathbf{F} \cdot d\mathbf{S}$$

where S is a surface of your choice with boundary ∂S coinciding with the curve C. Hence show that Stokes' theorem is satisfied.

7. In this question you are asked to prove Green's theorem on a rectangle directly. Let $\mathbf{v}(\mathbf{r}) = f(x,y)\hat{\mathbf{x}} + g(x,y)\hat{\mathbf{y}}$ be a vector field that lies in the (x,y)-plane and depends only on x and y. Without using Stokes' theorem, show that

$$\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx dy = \int_{\partial D} f\, dx + g\, dy,$$

where $D = \{(x,y) \,|\, a < x < b,\ c < y < d\}$ and ∂D is its boundary.

8. Show that for any two scalar fields $f(\mathbf{r}), g(\mathbf{r})$ around a closed curve C

$$\int_C (f\boldsymbol{\nabla} g + g\boldsymbol{\nabla} f) \cdot d\mathbf{r} = 0$$