MATH20901 Multivariable Calculus: Problems 5 [1]

1. Verify Stokes' theorem for the (hemispheric) surface $|\mathbf{r}| = 3$, $z \geq 0$ and the vector field $\mathbf{v}(\mathbf{r}) = (y, -x, 0)$.

2. If there exists a scalar function $f(x, y)$ such that a vector field $\mathbf{F} = \boldsymbol{\nabla} f$ show, using Green's theorem in the plane, that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ where C is a closed curve.

3. Deduce the value of integral in Problem Sheet 4 Q4 when the square is rotated in the (x, y)-plane by an arbitrary angle about the vertex at $(0, 0)$.

4. (a) Compute
$$\int_S \mathbf{F} \cdot d\mathbf{S}$$
when $\mathbf{F} = (x, y, z^4)$ and S is the conical surface formed by $z = \sqrt{x^2 + y^2}$ for $0 < z < 1$; the unit normal to the surface is directed away from the axis of the cone.

   (b) Find
$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F} \, dV$$
where V is the volume inside the cone defined above.

   (c) Without doing any further computations, deduce
$$\int_{x^2+y^2<1, z=1} \mathbf{F} \cdot d\mathbf{S}.$$
Check your answer.

5. Evaluate $\int_{\partial V} \mathbf{F} \cdot d\mathbf{S}$, where $\mathbf{F}(x, y, z) = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} - z\hat{\mathbf{z}}$ and V is the unit cube in the first octant. Perform the calculation directly and check by using the divergence theorem.

6. On Problem Sheet 2, Q4(b) it was shown that $\Delta\left(\frac{1}{r}\right) = 0$ for $r \neq 0$. Let V be a sphere of arbitrary non-zero radius centred on the origin and S its enclosing surface. Use the divergence theorem to show that
$$\int_V \Delta\left(\frac{1}{r}\right) dV = -4\pi.$$
How do you explain this ?

7. Guass' Law relates the flux of an electric field $\mathbf{E}(\mathbf{r})$ through a surface S to the density of charges in V, within S, $\rho(\mathbf{r})$ via
$$\int_S \mathbf{E} \cdot d\mathbf{S} = 4\pi \int_V \rho \, dV = Q$$
where Q is the net charge.

(a) Show that (one of Maxwell's equations)

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \qquad \text{at all points in space}$$

(b) Compute Q when $\mathbf{E} = (x, y, z)$ and V is the cuboid with vertices $(\pm 1, \pm 1, \pm 1)$. (Hint: use the easiest method you can).

8. We are all happy with the idea that volume of a region $V \subset \mathbb{R}^3$ can be defined as $\int_V dx\,dy\,dz$. Show that an alternative way of calculating it is

$$\frac{1}{3}\int_S \mathbf{r} \cdot \hat{\mathbf{n}}\, dS$$

where S is the surface enclosing V and $\hat{\mathbf{n}}$ the unit outward normal to S.

Confirm your answer by applying the new formula directly to a sphere of radius a.

9. In a style similar to Q8, can you think of a way of computing surface areas of 3D objects using integrals over the volume ?

10. Use Green's identity Identity to show, for two scalar fields $u(\mathbf{r})$, $v(\mathbf{r})$ that, if $\Delta u = 0$ in a volume V bounded by a surface S, upon which $v = 0$,

$$\int_V \nabla u \cdot \nabla v\, dV = 0$$