UNIVERSITY OF BRISTOL

Examination for the Degree of B.Sc. and M.Sci. (Level 5)

MULTIVARIABLE CALCULUS

MATH 20901

(Paper Code MATH-20901)

April 2013, 1 hours 30 minutes

This paper contains **three** *questions. A candidate's best TWO answers used for assessment.*

Calculators are **not** *permitted in this examination.*

Do not turn over until instructed.

1. (Total marks: 25)

   (a) i. (3 marks) Let $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$. Define what it means that $\mathbf{F}$ is differentiable at a point $\mathbf{x} \in \mathbb{R}^m$.

   ii. (3 marks) Let $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$ be differentiable at $\mathbf{x} \in \mathbb{R}^m$, and let $a \in \mathbb{R}$. Prove that $a\mathbf{F}$ is differentiable at $\mathbf{x}$.

   iii. (3 marks) Give an example of two functions $\mathbf{F}, \mathbf{G} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathbf{F} \neq \mathbf{G}$, but $\mathbf{F}'(0) = \mathbf{G}'(0)$.

   (b) (4 marks) Consider the transformation $(x, y) = (r\cos\theta, r\sin\theta)$. Show that the Jacobian determinant is given by

   $$\frac{\partial(x, y)}{\partial(r, \theta)} = r.$$

   When can we solve for (r, θ) in terms of (x, y)?

   (c) (12 marks) Show that the pair of equations

   $$x^2 - y^2 - u^3 + v^2 + 4 = 0, \quad 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0$$

   determine local functions $x(u, v)$ and $y(u, v)$ defined for (u, v) near $u = 2$ and $v = 1$ such that $x(2, 1) = 2$ and $y(2, 1) = -1$. Compute $\dfrac{\partial u}{\partial x}$ at $(2, 1)$.

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2. (Total marks: 25)

(a) (4 marks) Let $\mathbf{v}(x, y, z)$ be the vector field on $\mathbb{R}^3$ given by

$$\mathbf{u}(x, y, z) = (x \sin z, yz, \cos z).$$

Compute $\boldsymbol{\nabla} \cdot \mathbf{u}$ and $\boldsymbol{\nabla} \times \mathbf{u}$.

(b) Cylindrical coordinates are defined by

$$\mathbf{r}(r, \theta, z) = (r \cos \theta, r \sin \theta, z).$$

i. (3 marks) Calculate the basis vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\mathbf{z}}$.

ii. (1 mark) Calculate the scale factors h_r, h_θ, and h_z.

iii. (7 marks) Calculate $(\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}$ in cylindrical coordinates, where $\mathbf{u} = u_r \hat{\mathbf{r}} + u_\theta \hat{\boldsymbol{\theta}} + u_z \hat{\mathbf{z}}$.

(c) i. (6 marks) Let $f(r)$ be a smooth scalar-valued function of $r = |\mathbf{r}|$, and let $\mathbf{a} \in \mathbb{R}^3$ be a constant vector. Calculate

$$\boldsymbol{\nabla} \times (\mathbf{r} \times \mathbf{a} f(r)) .$$

ii. (4 marks) Let $\mathbf{u}$ be a vector field in $C(\mathbb{R}^3, \mathbb{R}^3)$. Show that

$$\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}) = \frac{1}{2} \boldsymbol{\nabla}(\mathbf{u} \cdot \mathbf{u}) - (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}.$$

Continued...

3. (Total marks: 25)

   (a) (4 marks) Let $\mathbf{v}(x, y) = (y^2, -xy)$ be a vector field on $\mathbb{R}^2$, and let C be the part of the circle $x^2 + y^2 = 1$ that starts at $(1, 0)$ and ends at $(0, 1)$, oriented clockwise. Compute $\int_C \mathbf{v} \cdot \mathbf{dr}$.

   (b) (4 marks) Let S be the surface of the unit sphere, and let $\mathbf{F} = F_r\hat{\mathbf{r}}$ be a vector field, where $\hat{\mathbf{r}} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ is the unit vector pointing radially outward from the sphere, where $0 \leq \phi \leq \pi$, and $0 \leq \theta \leq 2\pi$. Show that

$$\int_S \mathbf{F} \cdot \mathbf{dS} = \int_0^{2\pi} \int_0^{\pi} F_r \sin\phi d\phi d\theta.$$

   (c) Let S be the surface in $\mathbb{R}^3$ given by the cone $z = (x^2 + y^2)^{1/2}$, $x^2 + y^2 \leq 1$. Let $\mathbf{u}(x, y, z)$ be the vector field given by

$$\mathbf{v}(x, y, z) = (z^2, z, y^2).$$

      i. (8 marks) Without using Stokes' theorem, compute $\int_S (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathbf{dS}$, where $\mathbf{dS}$ is oriented inward toward the z-axis.

      ii. (5 marks) Recalculate the surface integral from (i) by computing the line integral along an appropriate path.

   (d) (4 marks) Let V be as in Gauss' theorem, and $f, g \in C^2(\mathbb{R}^3, \mathbb{R})$. Prove that

$$\int_{\partial V} f\boldsymbol{\nabla} g \cdot \mathbf{n} dS = \int_V (f \triangle g + \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g)\, dV.$$

End of examination.