UNIVERSITY OF BRISTOL

Examination for the Degree of B.Sc. and M.Sci. (Level 5)

**MULTIVARIABLE CALCULUS**

MATH 20901

(Paper Code MATH-20901)

---

January 2014, 1 hours 30 minutes

---

*This paper contains* **three** *questions. A candidate's best TWO answers used for assessment.*

*Calculators are* **not** *permitted in this examination.*

*On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.*

*Do not turn over until instructed.*

1. (Total marks: 25)

   (a) i. (3 marks) Let $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$. Define what it means that $\mathbf{F}$ is differentiable at a point $\mathbf{x} \in \mathbb{R}^m$.

   ii. (5 marks) Let

   $$\mathbf{f}(u, v, w) = (v^2 + uw, u^2 + w^2, u^2 v - w^3),$$

   and

   $$\mathbf{g}(x, y) = (xy^3, x^2 - y^2, 3x + 5y),$$

   and define $\mathbf{H}(x, y) = (f \circ g)(x, y)$. Compute $\mathbf{H}'(-1, 1)$.

   (b) (4 marks) Given

   $$z = f\left(\frac{x+y}{x-y}\right),$$

   where $f \in C^1(\mathbb{R}^2, \mathbb{R})$, show that

   $$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

   (c) Define the function $f \colon \mathbb{R}^2 \to \mathbb{R}$ by

   $$f(\mathbf{x}) = \begin{cases} \frac{x_1 x_2}{\sqrt{x_1^2 + x_2^2}}, & \mathbf{x} \neq 0, \\ 0, & \mathbf{x} = 0. \end{cases}$$

   i. (5 marks) Show that all partial derivatives exist for $\mathbf{x} \in \mathbb{R}^2$, and calculate them.

   ii. (3 marks ) Show that the partial derivatives of $f$ are not continuous at the origin.

   iii. (5 marks) Show that $f$ is not differentiable at the origin.

*Continued...*

2. (Total marks: 25)

(a) Simplify the following expressions in $\mathbb{R}^3$; summation convention applies.

i. (1 mark) $\delta_{jj}^2$,

ii. (1 mark) $\epsilon_{ijj}$,

iii. (3 marks) $\delta_{ij}\delta_{jk}\epsilon_{ilm}\epsilon_{lkn}$.

(b) Evaluate the following for $\mathbf{r} \in \mathbb{R}^3$, $r = |\mathbf{r}|$, where $\mathbf{a}$ is a constant vector. Use summation convention.

i. (3 marks)

$$\boldsymbol{\nabla} \times (\mathbf{a} \times \mathbf{r}),$$

ii. (3 marks)

$$\boldsymbol{\nabla} \cdot \left(\frac{\mathbf{r}}{r^2}\right).$$

(c) Spherical coordinates are defined by

$$\mathbf{r}(r, \phi, \theta) = (r \sin\phi \cos\theta, r \sin\phi \sin\theta, r \cos\phi).$$

i. (4 marks) Calculate the basis vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{\theta}}$, as well as the scale factors $h_r$, $h_\phi$, and $h_\theta$.

ii. (5 marks) Calculate $\boldsymbol{\nabla} \cdot \mathbf{u}$ in spherical coordinates, where $\mathbf{u} = \sin\theta\, \hat{\mathbf{r}}$.

(d) (5 marks) Let $\mathbf{u}, \mathbf{v}$ be vector fields in $C(\mathbb{R}^3, \mathbb{R}^3)$. Show, using summation convention, that

$$\boldsymbol{\nabla} \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot (\boldsymbol{\nabla} \times \mathbf{u}) - \mathbf{u} \cdot (\boldsymbol{\nabla} \times \mathbf{v}).$$

*Continued...*

3. (Total marks: 25)

   (a) (5 marks) Let $f(\mathbf{r})$ be a scalar field and let $C$ be a curve in $\mathbb{R}^3$ parameterised by the path $\mathbf{p}(s)$, $a \leq s \leq b$. Prove that

   $$\int_C \boldsymbol{\nabla} f \cdot d\mathbf{r} = f(\mathbf{p}(b)) - f(\mathbf{p}(a)).$$

   (b) Consider a sphere of radius $R$ about the origin, parameterized by

   $$\mathbf{s}(\phi, \theta) = R(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),$$

   with the vector field $\mathbf{f} = (y, -x, z)$.

   i. (4 marks) Calculate the line integral

   $$\int_C \mathbf{f} \cdot d\mathbf{r},$$

   where $C$ is the closed path along the equator going west (by convention, $\phi = 0$ is the north pole).

   ii. (5 marks) Verify Stokes' theorem by integrating a suitable vector field over the northern hemisphere.

   iii. (1 mark) What would the orientation of the surface be if the integration in ii. had been over the southern hemisphere?

   (c) Let $V$ be as in Gauss' theorem, where $\partial V$ is oriented such that its normal points to the outside of $V$.

   i. (5 marks) Let $f, g \in C^2(\mathbb{R}^3, \mathbb{R})$. Prove that

   $$\int_{\partial V} (f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f) \cdot \mathbf{n} dS = \int_V (f\triangle g - g\triangle f)\, dV.$$

   ii. (5 marks) Let $\phi$ be a scalar field in $C(\mathbb{R}^3, \mathbb{R})$. Use Gauss' theorem to show that

   $$\int_V \boldsymbol{\nabla}\phi dV = \int_{\partial V} \phi\mathbf{n} dS.$$

*End of examination.*