EXAMINATION SOLUTIONS

Multivariable Calculus
April 2013

1. (25 marks.)

(a) i. (3 marks) Let $\mathbf{h} \in \mathbb{R}^m$ and

$$\mathbf{r} = \mathbf{F}(\mathbf{x} + \mathbf{h}) - \mathbf{F}(\mathbf{x}) - A\mathbf{h}$$

There exists an $A \in \mathbb{R}^{n\times m}$ such that

$$\lim_{\mathbf{h}\to 0} \frac{||\mathbf{r}||}{||\mathbf{h}||} = 0.$$

ii. (3 marks) Let $A = \mathbf{F}'(\mathbf{x})$ and define

$$\tilde{\mathbf{r}} = a\mathbf{F}(\mathbf{x} + \mathbf{h}) - a\mathbf{F}(\mathbf{x}) - aA\mathbf{h}.$$

Then $\tilde{\mathbf{r}} = a\mathbf{r}$, and thus

$$\lim_{\mathbf{h}\to 0} \frac{||\tilde{\mathbf{r}}||}{||\mathbf{h}||} = a \lim_{\mathbf{h}\to 0} \frac{||\mathbf{r}||}{||\mathbf{h}||} = 0.$$

iii. (3 marks) Let $\mathbf{F} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\mathbf{G} = \begin{pmatrix} x_1^2 \\ 0 \end{pmatrix}$. Then $\mathbf{F}' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $\mathbf{G}' = \begin{pmatrix} 2x_1 & 0 \\ 0 & 0 \end{pmatrix}$, so $\mathbf{F}'(0) = \mathbf{G}'(0)$, but $\mathbf{F} \neq \mathbf{G}$.

(b) (4 marks)

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

We can solve for (r,θ) in terms of (x,y) whenever $r \neq 0$.

(c) (12 marks) We define $\mathbf{F} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ as

$$F_1(u,v,x,y) = x^2 - y^2 - u^3 + v^2 + 4, \quad F_2(u,v,x,y) = 2xy + y^2 - 2u^2 + 3v^4 + 8.$$

Then $F_1(2,1,2,-1) = F_2(2,1,2,-1) = 0$ and

$$\frac{\partial F_1}{\partial u} = -3u^2,\ \frac{\partial F_1}{\partial v} = 2v,\ \frac{\partial F_1}{\partial x} = 2x,\ \frac{\partial F_1}{\partial y} = -2y$$
$$\frac{\partial F_2}{\partial u} = -4u,\ \frac{\partial F_2}{\partial v} = 12v^3,\ \frac{\partial F_2}{\partial x} = 2y,\ \frac{\partial F_2}{\partial y} = 2x + 2y,$$

so that at $(x,y) = (2,-1)$ and $(u,v) = (2,1)$

$$\frac{\partial F_1}{\partial u} = -12,\ \frac{\partial F_1}{\partial v} = 2,\ \frac{\partial F_1}{\partial x} = 4,\ \frac{\partial F_1}{\partial y} = 2$$
$$\frac{\partial F_2}{\partial u} = -8,\ \frac{\partial F_2}{\partial v} = 12,\ \frac{\partial F_2}{\partial x} = -2,\ \frac{\partial F_2}{\partial y} = 2.$$

Now

$$\begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ -2 & 2 \end{vmatrix} = 12 \neq 0,$$

and according to the Implicit function theorem, this guarantees the existence of local functions $x(u, v)$ and $y(u, v)$ near $(u, v) = (2, 1)$. Now by partial differentiation

$$\begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix} + \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = 0,$$

or

$$\begin{pmatrix} -12 & 2 \\ -8 & 12 \end{pmatrix} \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = - \begin{pmatrix} 4 & 2 \\ -2 & 2 \end{pmatrix}.$$

But

$$\begin{pmatrix} -12 & 2 \\ -8 & 12 \end{pmatrix}^{-1} = \begin{pmatrix} -\frac{3}{32} & \frac{1}{64} \\ -\frac{1}{16} & \frac{3}{32} \end{pmatrix},$$

and so

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} -\frac{3}{32} & \frac{1}{64} \\ -\frac{1}{16} & \frac{3}{32} \end{pmatrix} \begin{pmatrix} 4 & 2 \\ -2 & 2 \end{pmatrix},$$

which means that $\dfrac{\partial u}{\partial x} = \dfrac{13}{32}$.

Continued...

2. (25 marks)

(a) (6 marks)

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \sin z + z - \sin z = z$$

and

$$\boldsymbol{\nabla}\times\mathbf{u} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x\sin z & yz & \cos z \end{vmatrix} = y\hat{\mathbf{x}} + x\cos z\hat{\mathbf{y}}.$$

(b) i. (3 marks)

$$\frac{\partial\mathbf{r}}{\partial r} = (\cos\theta, \sin\theta, 0) = \hat{\mathbf{r}},$$
$$\frac{\partial\mathbf{r}}{\partial\theta} = r(-\sin\theta, \cos\theta, 0) = r\hat{\boldsymbol{\theta}},$$
$$\frac{\partial\mathbf{r}}{\partial z} = (0,0,1) = \hat{\mathbf{z}}.$$

ii. (1 mark) From the preceding,

$$h_r = 1, \quad h_\theta = r, \quad h_z = 1.$$

iii. (7 marks)

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

and $\mathbf{u} = u_r\hat{\mathbf{r}} + u_\theta\hat{\boldsymbol{\theta}} + u_z\hat{\mathbf{z}}$, so that

$$\mathbf{u}\cdot\nabla = u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}.$$

Now

$$\frac{\partial\hat{\mathbf{r}}}{\partial\theta} = \hat{\boldsymbol{\theta}}, \quad \frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} = -\hat{\mathbf{r}},$$

and so

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = u_r\left(\frac{\partial u_r}{\partial r}\hat{\mathbf{r}} + \frac{\partial u_\theta}{\partial r}\hat{\boldsymbol{\theta}} + \frac{\partial u_z}{\partial r}\hat{\mathbf{z}}\right) + \frac{u_\theta}{r}\left(\frac{\partial u_r}{\partial\theta}\hat{\mathbf{r}} + \frac{\partial u_\theta}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{\partial u_z}{\partial\theta}\hat{\mathbf{z}} + u_r\hat{\boldsymbol{\theta}} - u_\theta\hat{\mathbf{r}}\right) +$$
$$u_z\left(\frac{\partial u_r}{\partial z}\hat{\mathbf{r}} + \frac{\partial u_\theta}{\partial z}\hat{\boldsymbol{\theta}} + \frac{\partial u_z}{\partial z}\hat{\mathbf{z}}\right) = \left(u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta^2}{r} + u_z\frac{\partial u_r}{\partial z}\right)\hat{\mathbf{r}} +$$
$$\left(u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r u_\theta}{r} + u_z\frac{\partial u_\theta}{\partial z}\right)\hat{\boldsymbol{\theta}} + \left(u_r\frac{\partial u_z}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_z}{\partial\theta} + u_z\frac{\partial u_z}{\partial z}\right)\hat{\mathbf{z}}.$$

(c) i. (6 marks)

$$(\boldsymbol{\nabla}\times(\mathbf{r}\times\mathbf{a}f))_i = \epsilon_{ijk}\partial_j\epsilon_{klm}r_l a_m f = \epsilon_{kij}\epsilon_{klm}\left(\delta_{lj}a_m f + r_l a_m\frac{r_j}{r}f'\right) =$$
$$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\left(\delta_{lj}a_m f + r_l a_m\frac{r_j}{r}f'\right) = a_i f - \delta_{jj}a_i f + r_i a_j\frac{r_j}{r}f' - a_i\frac{r_j^2}{r}f' =$$
$$\left(-\mathbf{a}(2f + rf') + \mathbf{r}\frac{\mathbf{a}\cdot\mathbf{r}}{r}f'\right)_i,$$

and so

$$\boldsymbol{\nabla}\times(\mathbf{r}\times\mathbf{a}f) = -\mathbf{a}(2f + rf') + \mathbf{r}\frac{\mathbf{a}\cdot\mathbf{r}}{r}f'.$$

ii. (4 marks)

$$(\mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u}))_i = \epsilon_{ijk}u_j\epsilon_{klm}\partial_l u_m = (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\,u_j\partial_l u_m =$$
$$u_m\partial_i u_m - u_l\partial_l u_i = \frac{1}{2}\partial_i u_m^2 - u_l\partial_l u_i = \left(\frac{1}{2}\boldsymbol{\nabla}\mathbf{u}^2-(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\right)_i .$$

Continued...

3. (25 marks)

(a) (4 marks) The circle is parameterized by $\mathbf{p}(s) = (\cos s, \sin s)$, from $s = 0$ to $s = -3\pi/2$. Further $\dfrac{d\mathbf{p}}{ds} = (-\sin s, \cos s)$ and $\mathbf{v} = (\sin^2 s, -\cos s \sin s)$. Then

$$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_0^{-3\pi/2} \left(-\sin^3 s - \cos^2 s \sin s\right) ds =$$
$$\sin^2 s \cos s\Big|_0^{-3\pi/2} - 3\int_0^{-3\pi/2} \cos^2 s \sin s ds = \cos^3 s\Big|_0^{-3\pi/2} = -1.$$

(b) (4 marks) The unit sphere is parameterized by

$$\mathbf{s}(\phi, \theta) = \hat{\mathbf{r}} = (\sin\phi\cos\theta, \sin\phi\sin\theta, cos\phi),$$

with $0 \le \phi \le \pi$, $0 \le \theta \le 2\pi$. Now

$$\mathbf{N} = \frac{\partial \mathbf{s}}{\partial \phi} \times \frac{\partial \mathbf{s}}{\partial \theta} = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi) \times (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0) =$$
$$(\sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta) =$$
$$\sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = \sin\phi\hat{\mathbf{r}}.$$

Thus

$$\int_S \mathbf{F}d\mathbf{S} = \int_0^{2\pi}\int_0^{\pi} \mathbf{F} \cdot \hat{\mathbf{r}} \sin\phi d\phi d\theta.$$

(c) i. (8 marks) The surface is parameterized by

$$\mathbf{s}(\phi, z) = (r\cos\phi, r\sin\phi, z) = (z\cos\phi, z\sin\phi, z),$$

where $(x^2 + y^2)^{1/2} = r = z$ on the surface of a cone, with $0 \le 2\pi$, $0 \le z \le 1$, so that

$$\mathbf{N} = \frac{\partial \mathbf{s}}{\partial z} \times \frac{\partial \mathbf{s}}{\partial \phi} = (\cos\phi, \sin\phi, 1) \times (-z\sin\phi, z\cos\phi, 0) = z(-\cos\phi, -\sin\phi, 1),$$

which points *inward*. The vector field is

$$\boldsymbol{\nabla} \times \mathbf{v} = (2y - x, 2z, z) = (2z\sin\phi - z\cos\phi, 2z, z),$$

and so

$$\int_S (\boldsymbol{\nabla} \times \mathbf{v})d\mathbf{S} = \int_0^1\int_0^{2\pi} (\boldsymbol{\nabla} \times \mathbf{v}) \cdot \mathbf{N} d\phi dz =$$
$$\int_0^1 z^2 dz \int_0^{2\pi} \left[-(2\sin\phi - \cos\phi)\cos\phi - 2\sin\phi + 1\right] d\phi =$$
$$\frac{1}{3}\left(\int_0^{2\pi} \cos^2\phi d\phi + 2\pi\right) = \pi.$$

ii. (5 marks) According to Stokes' theorem, the surface integral of (i) must equal $\int_{\partial S} \mathbf{v} \cdot \mathbf{dr}$, where ∂S is $z = 1$, $x^2 + y^2 = 1$, traversed in the clockwise direction: $\mathbf{p}(s) = (\cos s, \sin s, 1)$. Then $\dfrac{d\mathbf{p}}{ds} = (-\sin s, \cos s, 0)$, and $\mathbf{v} = (1, \cos s, \sin^2 s)$, so that

$$\int_{\partial S} \mathbf{v} \cdot \mathbf{dr} = \int_0^{2\pi} \mathbf{v} \cdot \frac{d\mathbf{p}}{ds} ds = \int_0^{2\pi} (-\sin s + \cos^2 s) ds = \pi,$$

as required.

(d) (4 marks) According to Gauss' theorem,

$$\int_V \boldsymbol{\nabla} (f \boldsymbol{\nabla} g)\, dV = \int_{\partial V} f \boldsymbol{\nabla} g \cdot \mathbf{n} dS,$$

and

$$\boldsymbol{\nabla} (f \boldsymbol{\nabla} g) = \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + f \boldsymbol{\nabla} \cdot \boldsymbol{\nabla} g = \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + f \triangle g.$$

End of solutions.