EXAMINATION SOLUTIONS

Multivariable Calculus
January 2014

1. (25 marks.)

(a) i. (3 marks) Let $\mathbf{h} \in \mathbb{R}^m$ and

$$\mathbf{r} = \mathbf{F}(\mathbf{x} + \mathbf{h}) - \mathbf{F}(\mathbf{x}) - A\mathbf{h}$$

There exists an $A \in \mathbb{R}^{n\times m}$ such that

$$\lim_{\mathbf{h}\to 0} \frac{||\mathbf{r}||}{||\mathbf{h}||} = 0.$$

ii. (5 marks) According to the chain rule,

$$\mathbf{H}'(-1,1) = \mathbf{f}'(\mathbf{g}(-1,1)) \circ \mathbf{g}'(-1,1).$$

Now $\mathbf{g}(-1,1) = (-1,0,2) \equiv (u,v,w)$. Thus

$$\mathbf{f}' = \begin{pmatrix} w & 2v & u \\ 2u & 0 & 2w \\ 2uv & u^2 & -3w^2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & -1 \\ -2 & 0 & 4 \\ 0 & 1 & -12 \end{pmatrix},$$

and

$$\mathbf{g}' = \begin{pmatrix} y^3 & 3xy^2 \\ 2x & -2y \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -2 & -2 \\ 3 & 5 \end{pmatrix},$$

so that

$$\mathbf{H}' = \begin{pmatrix} 2 & 0 & -1 \\ -2 & 0 & 4 \\ 0 & 1 & -12 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -2 & -2 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} -1 & -11 \\ 10 & 26 \\ -38 & -62 \end{pmatrix}.$$

(b) (4 marks) We use the chain rule to compute

$$\frac{\partial z}{\partial x} = \left(\frac{1}{x-y} - \frac{x+y}{(x-y)^2}\right) f' = -2f'\frac{y}{(x-y)^2}$$

and

$$\frac{\partial z}{\partial y} = \left(\frac{1}{x-y} + \frac{x+y}{(x-y)^2}\right) f' = 2f'\frac{x}{(x-y)^2}.$$

But this means that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\left(-\frac{xy}{(x-y)^2} + \frac{xy}{(x-y)^2}\right) f' = 0,$$

as claimed.

(c) i. (5 marks) If $\mathbf{x} \neq 0$, the denominator is nonzero, and we can use the chain rule to find

$$\frac{\partial f}{\partial x_1} = \frac{x_2}{(x_1^2 + x_2^2)^{1/2}} - \frac{x_1^2 x_2}{(x_1^2 + x_2^2)^{1/2}} = \frac{x_2^3}{(x_1^2 + x_2^2)^{3/2}},$$

and

$$\frac{\partial f}{\partial x_2} = \frac{x_1^3}{(x_1^2 + x_2^2)^{3/2}}$$

by symmetry. For the case $\mathbf{x} = 0$, note that $f(t, 0) = 0$ for all $t \in \mathbb{R}$. Thus

$$\lim_{t \to 0} \frac{f(t, 0) - f(0, 0)}{t} = 0,$$

so that $\dfrac{\partial f}{\partial x_1} = 0$ at the origin. By symmetry, $\dfrac{\partial f}{\partial x_2} = 0$.

ii. (3 marks) Consider the path $(x_1, x_2) = (0, t)$ for $t \to 0$. According to i.,

$$\frac{\partial f}{\partial x_1} = \frac{t^3}{t^3} = 1,$$

which does not converge to 0 as $t \to 0$. Thus $\dfrac{\partial f}{\partial x_1}$ is not continuous at the origin, and the same is true for $\dfrac{\partial f}{\partial x_2}$ by symmetry.

iii. (5 marks) Assume that f were differentiable at the origin. Then

$$f'\mathbf{e}_1 = \frac{\partial f}{\partial x_1} = 0$$

and

$$f'\mathbf{e}_2 = \frac{\partial f}{\partial x_2} = 0,$$

and so $f' = (0, 0)$. Now by direct calculation, for $\mathbf{v} = (1, 1)$

$$D_{\mathbf{v}} f = \lim_{t \to 0} \frac{f(t, t)}{t} = \frac{t^2}{\sqrt{2}tt} = \frac{1}{\sqrt{2}}.$$

But on the other hand, since f is differentiable,

$$D_{\mathbf{v}} f = A(\mathbf{e}_1 + \mathbf{e}_2) = 0,$$

which is a contradiction.

Continued...

2. (25 marks)

   (a) i. (1 mark) $\delta_{jj}^2 = 1 + 1 + 1 = 3$,

   ii. (1 mark) $\epsilon_{ijj} = -\epsilon_{ijj}$, thus $\epsilon_{ijj} = 0$.

   iii. (3 marks)

$$\delta_{ij}\delta_{jk}\epsilon_{ilm}\epsilon_{lkn} = \epsilon_{jlm}\epsilon_{ljn} = -(\delta_{jj}\delta_{mn} - \delta_{jn}\delta_{mj}) = -(3\delta_{mn} - \delta_{mn}) = -2\delta_{mn}.$$

   (b) i. (3 marks)

$$[\nabla \times (\mathbf{a} \times \mathbf{r})]_i = \epsilon_{ijk}\partial_j\epsilon_{klm}a_l r_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})a_l\partial_j r_m =$$
$$(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})a_l\delta_{jm} = (\delta_{il}\delta_{jj} - \delta_{ij}\delta_{jl})a_l = 3a_i - a_j\delta_{ij} = 2a_i.$$

   Thus

$$\nabla \times (\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}.$$

   ii. (3 marks)

$$\nabla \cdot \left(\frac{\mathbf{r}}{r^2}\right) = \partial_i\left(\frac{r_i}{r^2}\right) = \frac{\delta_{ii}}{r^2} - 2\frac{r_i r_i}{r^4} = \frac{3}{r^2} - 2\frac{r^2}{r^4} = \frac{1}{r^2}.$$

   (c) i. (4 marks)

$$\frac{\partial \mathbf{r}}{\partial r} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = h_r\hat{\mathbf{r}};$$

   $h_r = |(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)| = 1$.

$$\frac{\partial \mathbf{r}}{\partial \phi} = r(\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi) = h_\phi\hat{\boldsymbol{\phi}};$$

   $h_\phi = |r(\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)| = r$.

$$\frac{\partial \mathbf{r}}{\partial \theta} = r(-\sin\phi\sin\theta, \sin\phi\cos\theta, 0) = h_\theta\hat{\boldsymbol{\theta}};$$

   $h_\theta = |r(-\sin\phi\sin\theta, \sin\phi\cos\theta, 0)| = r\sin\phi$.
   This means that

$$\hat{\mathbf{r}} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),$$
$$\hat{\boldsymbol{\phi}} = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi),$$
$$\hat{\boldsymbol{\theta}} = (-\sin\theta, \cos\theta, 0).$$

   ii. (5 marks)

$$\nabla = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\phi}}}{r}\frac{\partial}{\partial \phi} + \frac{\hat{\boldsymbol{\theta}}}{r\sin\phi}\frac{\partial}{\partial \theta},$$

   and $\mathbf{u} = \sin\theta\hat{\mathbf{r}}$, so that

$$\nabla \cdot \mathbf{u} = \hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\frac{\partial \sin\phi}{\partial r} + \sin\theta\frac{\hat{\boldsymbol{\phi}}}{r} \cdot \frac{\partial \hat{\mathbf{r}}}{\partial \phi} + \sin\theta\frac{\hat{\boldsymbol{\theta}}}{r\sin\phi} \cdot \frac{\partial \hat{\mathbf{r}}}{\partial \theta}.$$

Now

$$\frac{\partial \hat{\mathbf{r}}}{\partial \phi} = (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi) = \hat{\boldsymbol{\phi}}, \quad \frac{\partial \hat{\mathbf{r}}}{\partial \theta} = (-\sin\phi\sin\theta, \sin\phi\cos\theta, 0) = \sin\phi\hat{\boldsymbol{\theta}},$$

and so

$$\boldsymbol{\nabla}\cdot\mathbf{u} = \sin\theta\frac{\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}}}{r} + \sin\phi\sin\theta\frac{\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}}}{r\sin\phi} = 2\frac{\sin\theta}{r}.$$

(d) (5 marks)

$$\boldsymbol{\nabla}\cdot(\mathbf{u}\times\mathbf{v}) = \epsilon_{ijk}\partial_i(u_j v_k) = \epsilon_{ijk}(\partial_i u_j)v_k + \epsilon_{ijk}u_j\partial_i v_k =$$
$$v_k\epsilon_{kij}(\partial_i u_j) - u_j\epsilon_{jik}u_j\partial_i v_k = \mathbf{v}\cdot(\boldsymbol{\nabla}\times\mathbf{u}) - \mathbf{u}\cdot(\boldsymbol{\nabla}\times\mathbf{v}).$$

Continued...

3. (25 marks)

(a) (5 marks) From the definition of the line integral, we have that

$$\int_C \boldsymbol{\nabla} f \cdot d\mathbf{r} = \int_a^b \boldsymbol{\nabla} f(\mathbf{p}(s)) \cdot \frac{d\mathbf{p}}{ds}\, ds.$$

But from the Chain Rule it follows that

$$\frac{d}{ds} f(\mathbf{p}(s)) = \frac{\partial f}{\partial r_i}(\mathbf{p}(s)) \frac{dp_i}{ds}(s).$$

Therefore,

$$\int_C \boldsymbol{\nabla} f \cdot d\mathbf{r} = \int_a^b \frac{d}{ds} f(\mathbf{p}(s))\, ds = f(\mathbf{p}(b)) - f(\mathbf{p}(a)),$$

where the last equality follows from the Fundamental Theorem of Calculus.

(b) i. (4 marks) The equator is along $\phi = \pi/2$, and thus

$$\mathbf{p}(\theta) = R(\cos\theta, \sin\theta, 0),$$

where θ runs from 2π to 0 (going west). Along the path,

$$\mathbf{f} = R(\sin\theta, -\cos\theta, 0),$$

and

$$\frac{\partial \mathbf{p}}{\partial \theta} = R(-\sin\theta, \cos\theta, 0).$$

Thus

$$\int_C \mathbf{f} \cdot d\mathbf{r} = \int_{2\pi}^0 \mathbf{f} \cdot \frac{\partial \mathbf{p}}{\partial \theta} d\theta = -R^2 \int_0^{2\pi} (-\sin^2\theta - \cos^2\theta) d\theta = 2\pi R^2.$$

ii. (5 marks) Stokes' theorem reads

$$\int_C \mathbf{f} \cdot d\mathbf{r} = \int_S \boldsymbol{\nabla} \times \mathbf{f} \cdot d\mathbf{S}.$$

Using the right-hand rule, the orientation of S (the northern hemisphere) should have the normal pointing inward. Thus

$$\mathbf{N} = \frac{\partial \mathbf{s}}{\partial \theta} \times \frac{\partial \mathbf{s}}{\partial \phi} = R^2(-\sin\phi\sin\theta, \sin\phi\cos\theta, 0) \times (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi) =$$
$$-R^2 \sin\phi(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),$$

and

$$\boldsymbol{\nabla} \times \mathbf{f} = (0, 0, -2).$$

Thus

$$\int_S \boldsymbol{\nabla} \times \mathbf{f} \cdot d\mathbf{S} = 2R^2 \int_0^{\pi/2} \int_0^{2\pi} \sin\phi\cos\phi\, d\theta\, d\phi = 2\pi R^2 \int_0^{\pi/2} \sin 2\phi\, d\phi = \pi R^2(1 - \cos\pi) = 2\pi R^2,$$

as in i.

iii. (1 mark) The orientation is the opposite of ii., i.e. facing outward.

(c) i. (5 marks) Put

$$\mathbf{h} = f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f$$

in Gauss' theorem,

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{h} dV = \int_{\partial V} \mathbf{h} \cdot \mathbf{n} dS.$$

Now

$$\boldsymbol{\nabla} (f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f) = \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g + f\triangle g - \boldsymbol{\nabla} g \cdot \boldsymbol{\nabla} f - g\triangle f = f\triangle g - g\triangle f,$$

and the result follows.

ii. (5 marks) According to Gauss' theorem,

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{v} dV = \int_{\partial V} \mathbf{v} \cdot \mathbf{n} dS.$$

Put $\mathbf{v} = \mathbf{a}\phi$, where $\mathbf{a}$ is an arbitrary, constant vector. Then $\boldsymbol{\nabla} \cdot \mathbf{v} = \mathbf{a} \cdot \boldsymbol{\nabla}\phi$, so that

$$\mathbf{a} \cdot \int_V \boldsymbol{\nabla}\phi dV = \int_V \boldsymbol{\nabla} \cdot \mathbf{v} dV = \int_{\partial V} \mathbf{v} \cdot \mathbf{n} dS = \mathbf{a} \cdot \int_{\partial V} \phi \mathbf{n} dS.$$

But since this is true for any $\mathbf{a}$, the statement follows.

End of solutions.