We can apply the same idea to the curvilinear system, so that the same point P is given by $\mathbf{q} = q_1\hat{\mathbf{q}}_1 + q_2\hat{\mathbf{q}}_2 + q_3\hat{\mathbf{q}}_3$ in terms of a local basis at P, written

$$\hat{\mathbf{q}}_1 = \frac{1}{h_1}\frac{\partial \mathbf{r}}{\partial q_1}, \qquad \hat{\mathbf{q}}_2 = \frac{1}{h_2}\frac{\partial \mathbf{r}}{\partial q_2}, \qquad \hat{\mathbf{q}}_3 = \frac{1}{h_3}\frac{\partial \mathbf{r}}{\partial q_3}$$

and to ensure $\hat{\mathbf{q}}_i$ are unit vectors, we normalise by

$$h_i = \left|\frac{\partial \mathbf{r}}{\partial q_i}\right|,$$

which are called the *metric coefficients* or *scale factors*.

Note: the use of Greek indices in, for e.g.

$$\hat{\mathbf{q}}_\alpha = \frac{1}{h_\alpha}\frac{\partial \mathbf{r}}{\partial q_\alpha}$$

for $\alpha = 1, 2, 3$ indicates that the summation convention is *not* applied.

Remark: Is this always possible ? I.e. is there always a unique map from one system to another ? This is the same as asking if there is an inverse map. Thus (by the inverse function theorem) the answer lies in the Jacobian matrix of the map, given here by $\mathbf{r}'(\mathbf{q})$ which is the matrix with $h_\alpha\hat{\mathbf{q}}_\alpha$ as column vectors (for $\alpha = 1, 2, 3$). Thus the Jacobian determinant

$$J_{\mathbf{r}} = \frac{\partial(x, y, z)}{\partial(q_1, q_2, q_3)} \equiv \begin{vmatrix} h_1(\hat{\mathbf{q}}_1)_1 & h_2(\hat{\mathbf{q}}_2)_1 & h_3(\hat{\mathbf{q}}_3)_1 \\ h_1(\hat{\mathbf{q}}_1)_2 & h_2(\hat{\mathbf{q}}_2)_2 & h_3(\hat{\mathbf{q}}_3)_2 \\ h_1(\hat{\mathbf{q}}_1)_3 & h_2(\hat{\mathbf{q}}_2)_3 & h_3(\hat{\mathbf{q}}_3)_3 \end{vmatrix}$$

must be non-vanishing.

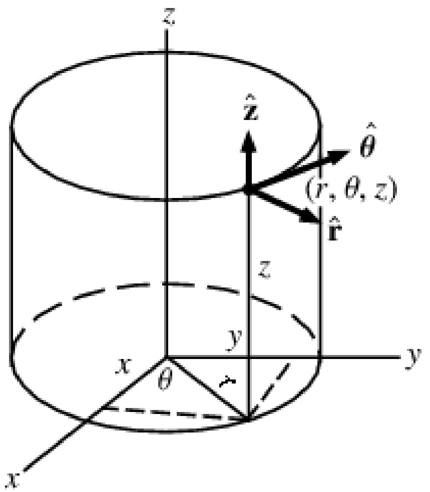

Figure 1: A local basis in cylindrical polar coordinates.

Defn: If local basis vectors of a curvilinear coordinate system are mutually orthogonal, we call it an *orthogonal* curvilinear coordinate system. Convention dictates that the system be *right-handed*, or $\hat{\mathbf{q}}_1 = \hat{\mathbf{q}}_2 \times \hat{\mathbf{q}}_3$.

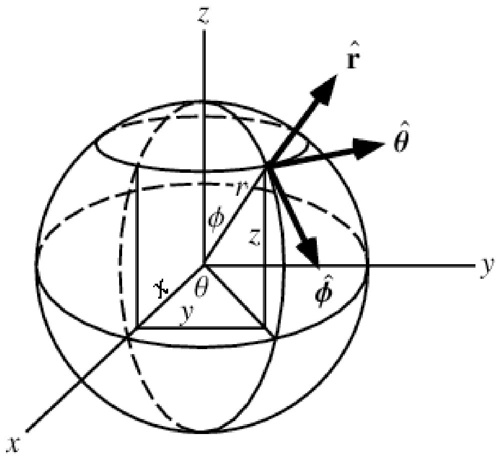

Figure 2: A local basis in spherical polar coordinates. The vector $\hat{\mathbf{r}}$ points along a ray from the center, $\hat{\boldsymbol{\phi}}$ points along the meridians, and $\hat{\boldsymbol{\theta}}$ along the parallels.

In the following, we will deal exclusively with orthogonal systems.

Examples:

1) Consider the linear transformation $x_i' = R_{ij}x_j$, where R is an orthogonal matrix (a matrix s.t. $R^T R = I$ which implies $R^{-1} = R^T$). I.e. the transformation can be written $\mathbf{x}' = R\mathbf{x}$. Then

$$(x_1, x_2, x_3) = \mathbf{r}(x_1', x_2', x_3') = R^T\mathbf{x}' = (R_{j1}x_j', R_{j2}x_j', R_{j3}x_j'),$$

and

$$\frac{\partial \mathbf{r}}{\partial x_1'} = (R_{11}, R_{12}, R_{13}),$$
$$\frac{\partial \mathbf{r}}{\partial x_2'} = (R_{21}, R_{22}, R_{23}),$$
$$\frac{\partial \mathbf{r}}{\partial x_3'} = (R_{31}, R_{32}, R_{33}),$$

The matrix equation $R^T R = I$ can be expressed as $\delta_{ik} = R^T_{ij}R_{jk} = R_{ji}R_{jk}$ and so the scale factors $h_\alpha = \sqrt{R^2_{j\alpha}} = \sqrt{\delta_{j\alpha}} = 1$.

Thus the local basis vectors are

$$\mathbf{e}_j{}' = (R_{j1}, R_{j2}, R_{j3}), \quad j = 1, 2, 3.$$

These are constant, i.e. they do not vary with position.

Note: $\mathbf{e}_j{}' \cdot \mathbf{e}_k{}' = R_{ji}R_{jk} = \delta_{ik}$ so the basis vectors are othonormal.

In other words, the new coordinate axes are a general rotation of the original $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ axes.

2) In 3D, *cylindrical polar coordinates* are defined by the mapping

$$(x, y, z) = \mathbf{r}(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$$

see Fig. 1. It follows that

$$\frac{\partial \mathbf{r}}{\partial r} = (\cos\theta, \sin\theta, 0), \quad \frac{\partial \mathbf{r}}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0), \quad \frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1).$$

The scale factors are

$$h_r = \left|\frac{\partial \mathbf{r}}{\partial r}\right| = 1, \quad h_\theta = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r, \quad h_z = \left|\frac{\partial \mathbf{r}}{\partial z}\right| = 1. \tag{11}$$

Thus the local basis vectors are (using standard notation):

$$\hat{\mathbf{r}} = (\cos\theta, \sin\theta, 0), \quad \hat{\boldsymbol{\theta}} = (-\sin\theta, \cos\theta, 0), \quad \hat{\mathbf{z}} = (0, 0, 1), \tag{12}$$

and these vary with position. Note that $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = \hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = \hat{\boldsymbol{\theta}} \cdot \hat{\mathbf{z}}$, and $\hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} \times \hat{\mathbf{z}} = 0$, so cylindrical coordinates are indeed orthogonal.

3) *Spherical polar coordinates* are defined by the mapping

$$(x, y, z) = \mathbf{r}(r, \phi, \theta) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi),$$

see Fig. 2. Now the derivatives with respect to the coordinates are

$$\begin{aligned}
\frac{\partial \mathbf{r}}{\partial r} &= (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), \\
\frac{\partial \mathbf{r}}{\partial \phi} &= (r\cos\phi\cos\theta, r\cos\phi\sin\theta, -r\sin\phi), \\
\frac{\partial \mathbf{r}}{\partial \theta} &= (-r\sin\phi\sin\theta, r\sin\phi\cos\theta, 0),
\end{aligned}$$

and the scale factors become (check):

$$h_r = \left|\frac{\partial \mathbf{r}}{\partial r}\right| = 1, \quad h_\phi = \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = r, \quad h_\theta = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = r\sin\phi. \tag{13}$$

Thus the local basis vectors are

$$\begin{aligned}
\hat{\mathbf{r}} &= (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), \\
\hat{\boldsymbol{\phi}} &= (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi), \\
\hat{\boldsymbol{\theta}} &= (-\sin\theta, \cos\theta, 0),
\end{aligned} \tag{14}$$

and vary with position. Again, $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\theta}} = 0$, and $\hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\theta}}$, so spherical coordinates are orthogonal.

2.8.2 Transformation of the gradient

The differential operator $\boldsymbol{\nabla}$ is the Cartesian vector

$$\boldsymbol{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right).$$

We want this to be transformed into derivatives w.r.t the local coordinates q_1, q_2, q_3. Then for fixed $\alpha = 1, 2, 3$, the chain rule gives

$$\frac{1}{h_\alpha}\frac{\partial f}{\partial q_\alpha} = \frac{1}{h_\alpha}\frac{\partial x_j}{\partial q_\alpha}\frac{\partial f}{\partial x_j} = \hat{\mathbf{q}}_\alpha \cdot \boldsymbol{\nabla} f \tag{15}$$

(summation over j is implied, but not α).

We are reminded that if $\mathbf{u} = u_1\hat{\mathbf{q}}_1 + u_2\hat{\mathbf{q}}_2 + u_3\hat{\mathbf{q}}_3$ then the orthonormal property of the local basis functions means $u_j = \mathbf{u}\cdot\hat{\mathbf{q}}_j$. If we compare with (15) with $\mathbf{u} = \boldsymbol{\nabla} f$ we get

$$\boldsymbol{\nabla} f = \sum_{\alpha=1}^{3}\frac{\hat{\mathbf{q}}_\alpha}{h_\alpha}\frac{\partial f}{\partial q_\alpha}, \qquad \text{and so} \qquad \boldsymbol{\nabla} = \sum_{\alpha=1}^{3}\frac{\hat{\mathbf{q}}_\alpha}{h_\alpha}\frac{\partial}{\partial q_\alpha}. \tag{16}$$

E.g. 1) In cylindrical polar coordinates, according to (16) and (11), we have

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

E.g. 2) In spherical coordinates, according to (16) and (3.2),

$$\boldsymbol{\nabla} = \hat{\mathbf{r}}\frac{\partial}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial\phi} + \hat{\boldsymbol{\theta}}\frac{1}{r\sin\phi}\frac{\partial}{\partial\theta}$$

2.8.3 Transformation of the divergence

To find $\boldsymbol{\nabla}\cdot\mathbf{u}$ in curvilinear coordinates we first need to express the vector field $\mathbf{u}$ in the local coordinate system. I.e.

$$\mathbf{u} = u_1\hat{\mathbf{q}}_1 + u_2\hat{\mathbf{q}}_2 + u_3\hat{\mathbf{q}}_3$$

The difficulty here is that both u_i and $\hat{\mathbf{q}}_i$ depend on (q_1, q_2, q_3). We come at the divergence in a slightly roundabout way.

First, we note from (16) that

$$\boldsymbol{\nabla} q_\alpha = \frac{\hat{\mathbf{q}}_\alpha}{h_\alpha}$$

Now note that

$$\boldsymbol{\nabla}\times(q_2\boldsymbol{\nabla} q_3) = q_2\underbrace{(\boldsymbol{\nabla}\times(\boldsymbol{\nabla} q_3))}_{=0} + (\boldsymbol{\nabla} q_2)\times(\boldsymbol{\nabla} q_3) = \frac{\hat{\mathbf{q}}_2}{h_2}\times\frac{\hat{\mathbf{q}}_3}{h_3} = \frac{\hat{\mathbf{q}}_1}{h_2 h_3}$$

Then from §2.6.1 (Null identites: $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{A}) = 0$, $\boldsymbol{\nabla}\times(\boldsymbol{\nabla} f) = 0$ for any $\mathbf{A}, f$,)

$$\boldsymbol{\nabla}\times\left(\frac{\hat{\mathbf{q}}_\alpha}{h_\alpha}\right) = 0, \qquad \boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{q}}_1}{h_2 h_3}\right) = 0.$$

Results true for the 2 cyclic permutations $(1\to 2, 2\to 3, 3\to 1)$

$$\boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{q}}_2}{h_1 h_3}\right) = \boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{q}}_3}{h_1 h_2}\right) = 0$$