So now

$$\begin{aligned}
\nabla\cdot\mathbf{u} &= \nabla.\left((u_1h_2h_3)\frac{\hat{\mathbf{q}}_1}{h_2h_3}\right)+2\text{ cyclic perms}\\
&= \frac{\hat{\mathbf{q}}_1}{h_2h_3}\cdot\nabla(u_1h_2h_3)+(u_1h_2h_3)\,\nabla\cdot\left(\frac{\hat{\mathbf{q}}_1}{h_2h_3}\right)+2\text{ cyclic perms}\\
&= \frac{\hat{\mathbf{q}}_1}{h_2h_3}\cdot\left(\sum_{\alpha=1}^{3}\frac{\hat{\mathbf{q}}_\alpha}{h_\alpha}\frac{\partial(u_1h_2h_3)}{\partial q_\alpha}\right)+2\text{ cyclic perms}\\
&= \frac{1}{h_1h_2h_3}\left[\frac{\partial(u_1h_2h_3)}{\partial q_1}+\frac{\partial(u_2h_1h_3)}{\partial q_2}+\frac{\partial(u_3h_1h_2)}{\partial q_3}\right]
\end{aligned}$$

using the fact that $\hat{\mathbf{q}}_\alpha\cdot\hat{\mathbf{q}}_\beta=\delta_{\alpha\beta}$.

E.g. (Cylindrical polar coordinates.) First write

$$\mathbf{u}=u_r\hat{\mathbf{r}}+u_\theta\hat{\boldsymbol{\theta}}+u_z\hat{\mathbf{z}}.$$

with $h_r=1$, $h_\theta=r$, $h_z=1$, so

$$\nabla\cdot\mathbf{u}=\frac{1}{r}\left[\frac{\partial(ru_r)}{\partial r}+\frac{\partial u_\theta}{\partial\theta}+\frac{\partial(ru_z)}{\partial z}\right]=\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{\partial u_z}{\partial z}$$

E.g. If $\mathbf{u}=f(r)\hat{\boldsymbol{\theta}}=(0,f(r),0)\equiv(u_r,u_\theta,u_z)$ then $\nabla\cdot\mathbf{u}=0$.

2.8.4 Transformation of curl

Similarly to div, we write

$$\begin{aligned}
\nabla\times\mathbf{u} &= \nabla\times\left((h_1u_1)\frac{\hat{\mathbf{q}}_1}{h_1}\right)+2\text{ cyclic perms}\\
&= \nabla(h_1u_1)\times\frac{\hat{\mathbf{q}}_1}{h_1}+(h_1u_1)\nabla\times\frac{\hat{\mathbf{q}}_1}{h_1}+2\text{ cyclic perms}\\
&= \left(\sum_{\alpha=1}^{3}\frac{\hat{\mathbf{q}}_\alpha}{h_\alpha}\frac{\partial(h_1u_1)}{\partial q_\alpha}\right)\times\frac{\hat{\mathbf{q}}_1}{h_1}+2\text{ cyclic perms}\\
&= \frac{\hat{\mathbf{q}}_2}{h_1h_3}\frac{\partial(h_1u_1)}{\partial q_3}-\frac{\hat{\mathbf{q}}_3}{h_1h_2}\frac{\partial(h_1u_1)}{\partial q_2}+2\text{ cyclic perms}
\end{aligned}$$

$$=\frac{\hat{\mathbf{q}}_1}{h_2h_3}\left(\frac{\partial(h_3u_3)}{\partial q_2}-\frac{\partial(h_2u_2)}{\partial q_3}\right)+\frac{\hat{\mathbf{q}}_2}{h_1h_3}\left(\frac{\partial(h_1u_1)}{\partial q_3}-\frac{\partial(h_3u_3)}{\partial q_1}\right)+\frac{\hat{\mathbf{q}}_3}{h_1h_2}\left(\frac{\partial(h_2u_2)}{\partial q_1}-\frac{\partial(h_1u_1)}{\partial q_2}\right)$$

2.9 Examples

1) The *Laplacian* of a scalar field ϕ is $\triangle\phi=\nabla\cdot\nabla\phi$. Since

$$\nabla\phi=\hat{\mathbf{r}}\frac{\partial\phi}{\partial r}+\hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial\phi}{\partial\theta}+\hat{\mathbf{z}}\frac{\partial\phi}{\partial z}.$$

we use the defn of div to give

$$\triangle\phi=\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial r}\right)+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)=\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\partial^2\phi}{\partial z^2}$$

2) Now the curl in cylindrical polars:

$$\boldsymbol{\nabla} \times \mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\hat{\mathbf{r}} + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\hat{\boldsymbol{\theta}} + \left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)\hat{\mathbf{z}},$$

Exercise: Do the same for spherical polars !!

Remark: If curvilinear system *not orthogonal* then a real mess.

3 Integration theorems of vector calculus

Having done differential vector calculus, we turn to integral vector calculus. These are equally important in applications as you will see in APDE2, Fluid Dynamics and beyond. We shall derive three (quite stunning) main integral identities all of which may be considered as higher-dimensional generalisations of the Fundamental Theorem of Calculus:

$$\int_a^b f'(x)\,dx = f(b) - f(a).$$

The LHS is a one-dimensional integral (i.e. an integral over a line) which is equated to zero-dimensional (i.e. pointwise) evaluations on the boundary of the integral (here at $x = a, b$).

Remark: The formula for integration by parts is found by letting $f(x) = u(x)v(x)$ in the above. Check !

3.1 The line integral of a scalar field

An ordinary 1D integral can be regarded as integration along a straight line. For example if $F(x)$ is the force on a particle alowed to move along the x-axis,

$$\int_{x_1}^{x_2} F(x)\,dx$$

is the "work done" moving it from x_1 to x_2. We want to integrals along general paths in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Defn: A *path* is a bijective (i.e. one-to-one) map $\mathbf{p} : [t_1, t_2] \to \mathbb{R}^3$ s.t. $t \mapsto \mathbf{p}(t)$. It connects the point $\mathbf{p}(t_1)$ to $\mathbf{p}(t_2)$ along a curve C, say. We say the curve C is *parameterised* by the path.

Defn: The line integral of a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ along a curve C is denoted

$$\int_C f(\mathbf{r})\,ds.$$

and $ds = |d\mathbf{r}|$ denotes the elemental arclength. Since $\mathbf{r} = \mathbf{p}(t)$ on C, $d\mathbf{r} = \mathbf{p}'(t)\,dt$ and so

$$\int_C f(\mathbf{r})\,ds = \int_{t_1}^{t_2} f(\mathbf{p}(t))|\mathbf{p}'(t)|\,dt.$$

E.g. 1) Let $\mathbf{p}(t) = (t, t, t)$ for $t \in [0, 1]$ connects the points $(0, 0, 0)$ to $(1, 1, 1)$ by a straight line of length $\sqrt{3}$. If $f = xyz$ then

$$\int_C f\,ds = \int_0^1 t^3\sqrt{1+1+1}\,dt = \frac{\sqrt{3}}{4}$$

E.g. 2) Let $\mathbf{p}(t) = (t^2, t^2, t^2)$ for $t \in [0, 1]$ parametrises the same curve as in 1). With the same f we have

$$\int_C f\,ds = \int_0^1 t^6\sqrt{(2t)^2 + (2t)^2 + (2t)^2}\,dt = \frac{\sqrt{3}}{4}.$$

Parameterisation is not unique. Suggests line integral independent of parametrisation. Easy to show (but not here).

Note: Line integral *is* dependent on path direction.

$$\int_C f\,ds = -\int_{-C} f\,ds$$

($-C$ is C in reverse). We are used to this notion in 1D, viz $\displaystyle\int_{x_1}^{x_2} f(x)dx = -\int_{x_2}^{x_1} f(x)dx$.

3.2 The line integral of a vector field

Defn: Let $\mathbf{F}(\mathbf{r}) : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field, and let $\mathbf{p}(t)$ be a path on the interval $[t_1, t_2]$. The *line integral of* $\mathbf{F}$ *along* $\mathbf{p}$ is defined by

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t_1}^{t_2} \mathbf{F}(\mathbf{p}(t))\cdot \mathbf{p}'(t)\,dt$$

as above.

Proposition: As above, the value of the line integral is not dependent on parametrisation of C but is negated by a reversal of C.

E.g: Integrate $\mathbf{F} = \sin\phi\hat{\mathbf{z}}$ along a meridian of a sphere of radius R from the south to the north pole.

A: From the description of the path, C, convenient to use spherical coordinates (r, ϕ, θ). I.e.

$$\mathbf{p}(\phi) = R\hat{\mathbf{r}} = R(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi);$$

(see earlier defn of $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\boldsymbol{\phi}}$ in spherical polars) then

$$\frac{d\mathbf{p}}{d\phi} = R\frac{\partial\hat{\mathbf{r}}}{\partial\phi} = R\hat{\boldsymbol{\phi}}.$$

From definition of $\hat{\boldsymbol{\phi}}$ in () we have $\hat{\mathbf{z}}\cdot\hat{\boldsymbol{\phi}} = -\sin\phi$ and so

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_\pi^0 \mathbf{F}\cdot\mathbf{p}'(\phi)\,d\phi = R\int_\pi^0 \sin\phi\,\hat{\mathbf{z}}\cdot\hat{\boldsymbol{\phi}}\,d\phi = -R\int_\pi^0 \sin^2\phi d\phi = \frac{R\pi}{2}.$$

Proposition: Let $f(\mathbf{r})$ be a scalar field and let C be a curve in $\mathbb{R}^3$ parameterised by the path $\mathbf{p}(t)$, $t_1 \le t \le t_2$. Then

$$\int_C \boldsymbol{\nabla} f\cdot d\mathbf{r} = f(\mathbf{p}(t_2)) - f(\mathbf{p}(t_1)).$$

This is the *fundamental theorem of Calculus for line integrals.*

Proof: We have

$$\int_C \boldsymbol{\nabla} f\cdot d\mathbf{r} = \int_{t_1}^{t_2} \boldsymbol{\nabla} f(\mathbf{p}(t))\cdot\mathbf{p}'(t)\,dt.$$

But from the Chain rule it follows that

$$\frac{d}{dt} f(\mathbf{p}(t)) = \mathbf{p}'(t) \cdot \boldsymbol{\nabla} f(\mathbf{p}(t))$$

Therefore,

$$\int_C \boldsymbol{\nabla} f \cdot d\mathbf{r} = \int_{t_1}^{t_2} \frac{d}{dt} f(\mathbf{p}(t))\, dt = f(\mathbf{p}(t_2)) - f(\mathbf{p}(t_1)),$$

from the Fundamental Theorem of Calculus.

Note: If C is closed, the line integral over a gradient field vanishes. As a result, line integrals of gradient fields are independent of the path C.

Remark: The line integral of a vector field is often called the *work integral*, since if $\mathbf{F}$ represents a force, the integral represents the work done moving a particle between two points. If $\mathbf{F} = \boldsymbol{\nabla} f$ for some scalar field f (often called the *potential*) then the work done moving the particle is independent of the path taken. Moreover the work done moving a particle which returns to the same position is zero. Such a force is called *conservative*.

E.g. The force due to gravity is $\mathbf{F} = (0, 0, -g) = \boldsymbol{\nabla} f$ if $f = -gz$ and so gravity is a conservative force.

3.3 Surface integrals of scalar and vector fields.

We now generalise 1D integrals to 2D integrals. We start with parametrisations of surfaces.

Defn: A path $\mathbf{p}(t)$, for $t \in [t_1, t_2]$ is *closed* if $\mathbf{p}(t_1) = \mathbf{p}(t_2)$. A closed path is *simple* it it does not intersect with itself apart from at the end points t_1, t_2.

Defn: Let $D \subset \mathbb{R}^2$, let ∂D represent the *boundary* of D (it should be a simple closed path) and let $\bar{D}$ be $D \cup \partial D$.

Now define a map $\mathbf{s} : \bar{D} \to \mathbb{R}^3$ s.t $(u, v) \mapsto \mathbf{s}(u, v)$ and $\partial \mathbf{s}/\partial u$, $\partial \mathbf{s}/\partial v$ are linearly independent on D. A *surface* $S \in \mathbb{R}^3$ is given in *parametrised form* by $S = \{\mathbf{s}(u, v) \,|\, (u, v) \in D\}$.

E.g. Let

$$D = \{(u, v) \,|\, u^2 + v^2 < R^2\}.$$

Then ∂D is the circle $\{(u, v) \,|\, u^2 + v^2 = R^2\}$ of radius R. Let $\mathbf{s} = (u, v, \sqrt{R^2 - u^2 - v^2})$, then S is a hemispherical surface.

Note: this is not the only way to parametrise a hemisphere; could use spherical polars.

Defn: The *integral of a scalar field f over a surface S* is denoted by

$$\int_S f(\mathbf{r})\, dS \equiv \int_S f(\mathbf{r})\, |d\mathbf{S}|$$

where $d\mathbf{S} = \hat{\mathbf{n}} dS$ and $\hat{\mathbf{n}}$ is a unit vector pointing out from S (a surface element is defined by its size dS and a direction, $\hat{\mathbf{n}}$, being the normal to the surface). Now the two vectors

$$\frac{\partial \mathbf{s}}{\partial u} du, \quad \frac{\partial \mathbf{s}}{\partial v} dv$$