But from the Chain rule it follows that

$$\frac{d}{dt} f(\mathbf{p}(t)) = \mathbf{p}'(t) \cdot \boldsymbol{\nabla} f(\mathbf{p}(t))$$

Therefore,

$$\int_C \boldsymbol{\nabla} f \cdot d\mathbf{r} = \int_{t_1}^{t_2} \frac{d}{dt} f(\mathbf{p}(t))\, dt = f(\mathbf{p}(t_2)) - f(\mathbf{p}(t_1)),$$

from the Fundamental Theorem of Calculus.

Note: If C is closed, the line integral over a gradient field vanishes. As a result, line integrals of gradient fields are independent of the path C.

Remark: The line integral of a vector field is often called the *work integral*, since if $\mathbf{F}$ represents a force, the integral represents the work done moving a particle between two points. If $\mathbf{F} = \boldsymbol{\nabla} f$ for some scalar field f (often called the *potential*) then the work done moving the particle is independent of the path taken. Moreover the work done moving a particle which returns to the same position is zero. Such a force is called *conservative*.

E.g. The force due to gravity is $\mathbf{F} = (0, 0, -g) = \boldsymbol{\nabla} f$ if $f = -gz$ and so gravity is a conservative force.

3.3 Surface integrals of scalar and vector fields.

We now generalise 1D integrals to 2D integrals. We start with parametrisations of surfaces.

Defn: A path $\mathbf{p}(t)$, for $t \in [t_1, t_2]$ is *closed* if $\mathbf{p}(t_1) = \mathbf{p}(t_2)$. A closed path is *simple* it it does not intersect with itself apart from at the end points t_1, t_2.

Defn: Let $D \subset \mathbb{R}^2$, let ∂D represent the *boundary* of D (it should be a simple closed path) and let $\bar{D}$ be $D \cup \partial D$.

Now define a map $\mathbf{s} : \bar{D} \to \mathbb{R}^3$ s.t $(u, v) \mapsto \mathbf{s}(u, v)$ and $\partial \mathbf{s}/\partial u$, $\partial \mathbf{s}/\partial v$ are linearly independent on D. A *surface* $S \in \mathbb{R}^3$ is given in *parametrised form* by $S = \{\mathbf{s}(u, v) \,|\, (u, v) \in D\}$.

E.g. Let

$$D = \{(u, v) \,|\, u^2 + v^2 < R^2\}.$$

Then ∂D is the circle $\{(u, v) \,|\, u^2 + v^2 = R^2\}$ of radius R. Let $\mathbf{s} = (u, v, \sqrt{R^2 - u^2 - v^2})$, then S is a hemispherical surface.

Note: this is not the only way to parametrise a hemisphere; could use spherical polars.

Defn: The *integral of a scalar field f over a surface S* is denoted by

$$\int_S f(\mathbf{r})\, dS \equiv \int_S f(\mathbf{r})\, |d\mathbf{S}|$$

where $d\mathbf{S} = \hat{\mathbf{n}} dS$ and $\hat{\mathbf{n}}$ is a unit vector pointing out from S (a surface element is defined by its size dS and a direction, $\hat{\mathbf{n}}$, being the normal to the surface). Now the two vectors

$$\frac{\partial \mathbf{s}}{\partial u} du, \quad \frac{\partial \mathbf{s}}{\partial v} dv$$

lie in the tangent plane to the surface S and so the area of the elemental rhomboid dS in the direction $\hat{\mathbf{n}}$ is, see Fig. **??**,

$$d\mathbf{S} = \mathbf{N}(u,v)dudv, \qquad \mathbf{N}(u,v) = \frac{\partial \mathbf{s}}{\partial u} \times \frac{\partial \mathbf{s}}{\partial v}$$

and so

$$\int_S f(\mathbf{r})\, dS = \int_D f(\mathbf{s}(u,v))|\mathbf{N}|\, du\, dv.$$

Note: If S lies in the 2D plane, then $\mathbf{s} = (x(u,v), y(u,v), 0)$ is a mapping from 2D to 2D and so

$$|\mathbf{N}| = \frac{\partial(x,y)}{\partial(u,v)}$$

is the Jacobian of the map (easy to confirm). We know this from Calculus 1 (e.g. $dxdy \mapsto rdrd\theta$).

Defn: Let $\mathbf{v}$ be a vector field on $\mathbb{R}^3$. The *integral of* $\mathbf{v}$ *over* S, is denoted

$$\int_S \mathbf{v}\cdot d\mathbf{S} \equiv \int_S \mathbf{v}\cdot\hat{\mathbf{n}}\, dS = \int_D \mathbf{v}(\mathbf{s}(u,v))\cdot\mathbf{N}(u,v)\, du\, dv,$$

as above.

Important remark: By analogy with line integrals, can show that the surface integral of a vector field is independent of parameterisation up to a sign. The sign depends on the orientation of the parameterisation, which is determined by the direction of the unit normal $\hat{\mathbf{n}} = \mathbf{N}/||\mathbf{N}||$. Thus, the direction of $\hat{\mathbf{n}}$, (or $\mathbf{N}$) *must be specified* in order to fix the sign of the integral unambiguously.

E.g.

Let S be the surface given by

$$S = \{(x,y,0)\,|\,x^2+y^2 \le 1\},$$

and let the vector field be given by

$$\mathbf{B}(\mathbf{r}) = r\hat{\mathbf{z}} + \hat{\mathbf{r}}.$$

Polar coordinates are a sensible parameterisation of S, i.e.

$$\mathbf{s}(r,\theta) = (r\cos\theta, r\sin\theta, 0),$$

so that $D = \{(r,\theta)\,|\,0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$. Then

$$\mathbf{N} = \frac{\partial \mathbf{s}}{\partial r} \times \frac{\partial \mathbf{s}}{\partial \theta} = r\hat{\mathbf{z}},$$

(which is the Jacobian in this 2D mapping, $\hat{\mathbf{z}}$ normal to the 2D plane) and

$$\int_S \mathbf{B}\cdot d\mathbf{S} = \int_0^{2\pi}\int_0^1 \mathbf{B}\cdot\hat{\mathbf{z}}\, rdrd\theta = \int_0^{2\pi}\int_0^1 r^2\, dr\, d\theta = \frac{2\pi}{3}$$

and we have used $\hat{\mathbf{r}}\cdot\hat{\mathbf{z}} = 0$.

3.4 Stokes' theorem

We now consider the case where the vector field $\mathbf{v}$ can be expressed as the curl of another vector field, i.e. $\mathbf{v} = \boldsymbol{\nabla} \times \mathbf{A}$. This frequently happens in applications.

Defn: The *boundary* of the surface S is denoted ∂S and, since it is mapped from the boundary ∂D, it inherits its properties, being a simple closed path. If $\mathbf{c}(t) \in \mathbb{R}^2$ is the simple closed path along ∂D then

$$\mathbf{p}(t) = \mathbf{s}(\mathbf{c}(t))$$

is the simple closed path along ∂S.

Important note: By convention, we choose an orientation of this curve according to the right-hand rule. That is, point the thumb of your right hand on ∂S in the direction of $\mathbf{N}$ (or $\hat{\mathbf{n}}$) and wrap your fingers round to indicate the sense of the integral.

Proposition: Let S be a surface in $\mathbb{R}^3$ with boundary ∂S; let $\mathbf{A}$ be a vector field. Then

$$\int_S \boldsymbol{\nabla} \times \mathbf{A} \cdot d\mathbf{S} = \int_{\partial S} \mathbf{A} \cdot d\mathbf{r}, \tag{17}$$

where $d\mathbf{S}$ and ∂S are consistently oriented according to the right-hand rule.

Before we give the outline proof of this important result, we give an example, which illustrates the theorem at work.

3.4.1 Example:

Consider the hemisphere of radius R, whose boundary is the equator of the sphere, of radius R. Let

$$\mathbf{f}(\mathbf{r}) = \boldsymbol{\omega} \times \mathbf{r},$$

where $\boldsymbol{\omega}$ is some fixed vector. On problem sheet 2, we have shown that $\boldsymbol{\nabla} \times \mathbf{f} = 2\boldsymbol{\omega} \equiv (2\omega_1, 2\omega_2, 2\omega_3)$.

Let us begin with the LHS hand side of (17). We calculate the surface integral using two different parameterizations:

(i) Cartesian coordinates. As in a previous example, the surface can be parametrised by the mapping $\mathbf{s} : D \to \mathbb{R}^3$ where $D = \{(u, v) \,|\, u^2 + v^2 < R^2\} \subset \mathbb{R}^2$ and

$$\mathbf{s}(u, v) = (u, v, \sqrt{R^2 - u^2 - v^2}).$$

We also have $\partial D = \{(u, v) \,|\, u^2 + v^2 = R^2\} \subset \mathbb{R}^2$, and its image is clearly $\partial S = \{(u, v, 0) \,|\, u^2 + v^2 = R^2\} \subset \mathbb{R}^3$.

Now

$$\mathbf{N}(u, v) = \frac{\partial \mathbf{s}}{\partial u} \times \frac{\partial \mathbf{s}}{\partial v} = \left(1, 0, -\frac{u}{w}\right) \times \left(0, 1, -\frac{v}{w}\right) = \left(\frac{u}{w}, \frac{v}{w}, 1\right),$$

writing $w = \sqrt{R^2 - u^2 - v^2}$, which is the z-coordinate on the sphere. Clearly, $\mathbf{N}$ points in the direction of the *outward* normal $\hat{\mathbf{r}} = (u, v, w)/R$. Then the surface integral is

$$\begin{aligned} \int_S \boldsymbol{\nabla} \times \mathbf{f} \cdot d\mathbf{S} &= \int_D \boldsymbol{\nabla} \times \mathbf{f} \cdot \mathbf{N}(u,v)\, dudv \\ &= \int_{u^2+v^2<R^2} \left(\frac{2\omega_1 u}{w} + \frac{2\omega_1 v}{w} + 2\omega_3 \right) dudv = \int_{u^2+v^2<R^2} 2\omega_3\, dudv \\ &= 2\pi\omega_3 R^2. \end{aligned}$$

(using the oddness of the first two components).

(ii) Using spherical polars, we parametrise the surface by

$$\mathbf{s}(\phi, \theta) = (R\sin\phi\cos\theta, R\sin\phi\sin\theta, R\cos\phi),$$

with $D = \{(\theta, \phi) \,|\, 0 \le \phi \le \pi/2,\ 0 \le \theta \le 2\pi\}$. So now

$$\mathbf{N} = \frac{\partial \mathbf{s}}{\partial \phi} \times \frac{\partial \mathbf{s}}{\partial \theta} = h_\phi h_\theta \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\theta}} = R^2 \sin\phi \hat{\mathbf{r}},$$

(see earlier definition) so that

$$\int_S \boldsymbol{\nabla} \times \mathbf{f} \cdot d\mathbf{S} = \int_D \boldsymbol{\nabla} \times \mathbf{f} \cdot R^2 \sin\phi \hat{\mathbf{r}}\, d\theta d\phi = 2R^2 \int_0^{\pi/2} \int_0^{2\pi} \sin\phi \boldsymbol{\omega} \cdot \hat{\mathbf{r}} d\theta d\phi,$$

but

$$\int_0^{2\pi} \boldsymbol{\omega} \cdot \hat{\mathbf{r}}\, d\theta = \int_0^{2\pi} (\omega_1 \sin\phi\cos\theta + \omega_2 \sin\phi\sin\theta + \omega_3\cos\phi)\, d\theta = 2\pi\omega_3\cos\phi.$$

Thus, taken together,

$$\int_S \boldsymbol{\nabla} \times \mathbf{f} \cdot d\mathbf{S} = 4\pi R^2 \omega_3 \int_0^{\pi/2} \sin\phi\cos\phi\, d\phi = 2\pi R^2 \omega_3,$$

the same as before.

Now the RHS, the integral around ∂S. A parameterisation is $\mathbf{p}(\theta) = (R\cos\theta, R\sin\theta, 0)$, $0 \le \theta < 2\pi$ and, according to $\mathbf{N}$ and the RH-thumb rule we integrate in an anticlockwise direction. Now

$$\int_{\partial S} \mathbf{f} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{f} \cdot \mathbf{p}'(\theta)\, d\theta,$$

and we have

$$\mathbf{p}'(\theta) == R(-\sin\theta, \cos\theta) = R\hat{\boldsymbol{\theta}},$$

(in cylindrical polars) and on ∂S, $\mathbf{p}(\theta) = R(\cos\theta, \sin\theta, 0) = R\hat{\mathbf{r}}$ so

$$\mathbf{f} \cdot \hat{\boldsymbol{\theta}} = R^2 (\boldsymbol{\omega} \times \hat{\mathbf{r}}) \cdot \hat{\boldsymbol{\theta}} = R^2 \boldsymbol{\omega} \cdot \left(\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} \right) = R^2 \boldsymbol{\omega} \cdot \hat{\mathbf{z}} = R^2 \omega_3.$$

using a vector triple product result. So now

$$\int_{\partial S} \mathbf{f} \cdot d\mathbf{r} = R^2 \omega_3 \int_0^{2\pi} d\theta = 2\pi R^2 \omega_3,$$

which confirms Stokes' theorem !!

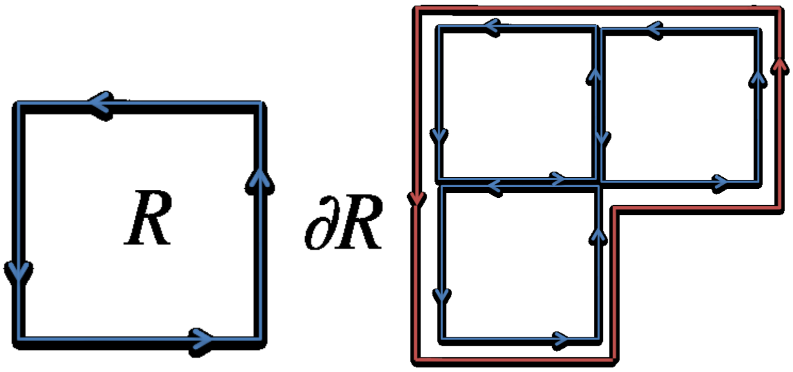

Figure 3: A number of rectangles (left) can be put together to cover the domain (right).

3.4.2 Outline proof of Stokes' theorem (non-exam)

We first prove for a rectangle in (u, v)-space. The loose argument then proceeds that rectangles can be assembled as a checkboard into larger domains, given that the limit of rectangle size can be taken to go to zero and since contributions from adjacent sides cancel leaving just the circuit around the total domain. See fig 3.
Let $D = \{(u, v) \,|\, 0 < u < a,\, 0 < v < b\}$. The surface S is defined by the map $\mathbf{s}(u, v) : D \to \mathbb{R}^3$
The boundary $\partial D = C_1 \cup C_2 \cup C_3 \cup C_4$ is mapped by $\mathbf{s}$ onto $\partial S = \partial S_1 \cup \partial S_2 \cup \partial S_3 \cup \partial S_4$.

For e.g. C_2 is the path $\mathbf{p}(t) = \mathbf{s}(a, t)$, $0 < t < b$ and

$$\int_{\partial S_2} \mathbf{A} \cdot d\mathbf{r} = \int_0^b \mathbf{A}(\mathbf{p}(t)) \cdot \mathbf{p}'(t)\, dt = \int_0^b \mathbf{A}(\mathbf{s}(a, v)) \cdot \frac{\partial \mathbf{s}}{\partial v}(a, v)\, dv$$

Similarly,

$$\int_{\partial S_4} \mathbf{A} \cdot d\mathbf{r} = -\int_0^b \mathbf{A}(\mathbf{s}(0, v)) \cdot \frac{\partial \mathbf{s}}{\partial v}(0, v)\, dv$$

(the minus sign accounts for reversing the orientation of the segment C_4, viz: $\int_b^0 = -\int_0^b$)

Combining results gives

$$\begin{aligned} \left(\int_{\partial S_2} + \int_{\partial S_4}\right) \mathbf{A}.d\mathbf{r} &= \int_0^b \left(\mathbf{A}(\mathbf{s}(a, v)) \cdot \frac{\partial \mathbf{s}}{\partial v}(a, v) - \mathbf{A}(\mathbf{s}(0, v)) \cdot \frac{\partial \mathbf{s}}{\partial v}(0, v)\right) dv \\ &= \int_0^b \int_0^a \frac{\partial}{\partial u}\left(\mathbf{A}(\mathbf{s}) \cdot \frac{\partial \mathbf{s}}{\partial v}\right) du\, dv \end{aligned}$$

by the FTC. We apply the same method to the side ∂S_1 and ∂S_3 and find

$$\left(\int_{\partial S_1} + \int_{\partial S_3}\right) \mathbf{A}.d\mathbf{r} = -\int_0^b \int_0^a \frac{\partial}{\partial v}\left(\mathbf{A}(\mathbf{s}) \cdot \frac{\partial \mathbf{s}}{\partial u}\right) du\, dv$$

and so

$$\int_{\partial S} \mathbf{A}.d\mathbf{r} = \int_D \left(\frac{\partial}{\partial u}\left(\mathbf{A}(\mathbf{s}) \cdot \frac{\partial \mathbf{s}}{\partial v}\right) - \frac{\partial}{\partial v}\left(\mathbf{A}(\mathbf{s}) \cdot \frac{\partial \mathbf{s}}{\partial u}\right)\right) dudv$$

Concentrate on the integrand of the LHS. So

$$\begin{aligned}
\frac{\partial}{\partial u}\left(\mathbf{A}(\mathbf{s})\cdot\frac{\partial \mathbf{s}}{\partial v}\right)-\frac{\partial}{\partial v}\left(\mathbf{A}(\mathbf{s})\cdot\frac{\partial \mathbf{s}}{\partial u}\right) &= \frac{\partial \mathbf{A}(\mathbf{s})}{\partial u}\cdot\frac{\partial \mathbf{s}}{\partial v}-\frac{\partial \mathbf{A}(\mathbf{s})}{\partial v}\cdot\frac{\partial \mathbf{s}}{\partial u} \\
&= \frac{\partial A_k}{\partial x_j}\left(\frac{\partial x_j}{\partial u}\frac{\partial x_k}{\partial v}-\frac{\partial x_j}{\partial v}\frac{\partial x_k}{\partial u}\right) \\
&= \frac{\partial A_k}{\partial x_j}\left(\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}\right)\frac{\partial x_l}{\partial u}\frac{\partial x_m}{\partial v} \\
&= \frac{\partial A_k}{\partial x_j}\epsilon_{ijk}\epsilon_{ilm}\frac{\partial x_l}{\partial u}\frac{\partial x_m}{\partial v} \\
&= \left(\epsilon_{ijk}\frac{\partial A_k}{\partial x_j}\right)\left(\epsilon_{ilm}\frac{\partial x_l}{\partial u}\frac{\partial x_m}{\partial v}\right) \\
&= (\boldsymbol{\nabla}\times\mathbf{A})\cdot\left(\frac{\partial \mathbf{s}}{\partial u}\times\frac{\partial \mathbf{s}}{\partial v}\right)=(\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{N}(u,v)
\end{aligned}$$

Hence we have

$$\int_{\partial S}\mathbf{A}.d\mathbf{r}=\int_D(\boldsymbol{\nabla}\times\mathbf{A})\cdot\mathbf{N}(u,v)\,dudv=\int_S(\boldsymbol{\nabla}\times\mathbf{A})\cdot d\mathbf{S}$$

as required. As mentioned earlier, we now “glue together” small rectangles to create the actual domain we wish to cover. This can be done formally.

3.4.3 Green’s theorem in the plane

If Stokes’ theorem is applied on a 2D plane (i.e. S is ‘flat’) w.l.o.g. $z=0$, then $d\mathbf{S}=\hat{\mathbf{z}}dS$ and $\mathbf{A}=(A_1(x,y),A_2(x,y),0)$ then we get

$$\int_S\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)dxdy=\int_{\partial S}\mathbf{A}\cdot d\mathbf{r}\equiv\int_{\partial S}A_1\,dx+A_2\,dy$$

and ∂S is anticlockwise by the RHT rule.

3.5 Volume integrals

Just as for integrals over lines and surfaces, for integrals over volumes there is an analogue of the Fundamental Theorem of Calculus, called Gauss’ theorem, or the divergence theorem.

3.5.1 Volume integrals of scalar fields

Let $f:\mathbb{R}^3\to\mathbb{R}$ s.t. $\mathbf{r}\mapsto f(\mathbf{r})$ be a scalar field. Let $V\subset\mathbb{R}^3$ have a boundary ∂V. If V is finite and simply connected (all points within V are connected to all other points by cts paths which lie entirely within V), then ∂V forms a *closed surface* (a surface with no boundaries). The volume integral of f is given by

$$\int_V f(\mathbf{r})\,dV\equiv\iiint f(x,y,z)\,dx\,dy\,dz$$

E.g.: if $f=1$, then $\int_V 1\,dV$ gives the physical volume of V.