Proposition: If we move to a different coordinate system, $\mathbf{q} = (q_1, q_2, q_3)$ from $\mathbf{r} = (x, y, z)$ under the mapping $\mathbf{r} : \mathbb{R}^3 \to \mathbb{R}^3$ s.t. $\mathbf{q} \mapsto \mathbf{r}(\mathbf{q})$ then

$$\int_V f(\mathbf{r})\, dxdydz = \int_{V_q} f(\mathbf{r}(\mathbf{q})) \left| \frac{\partial(x, y, z)}{\partial(q_1, q_2, q_3)} \right| dq_1 dq_2 dq_3$$

where V_q is mapped by $\mathbf{r}$ into V.

The scale factor is the Jacobian determinant of the mapping.

Proof: The elemental volume $dxdydz$ is $(\hat{\mathbf{z}}dz) \cdot ((\hat{\mathbf{x}}dx) \times (\hat{\mathbf{y}}dy))$. Under the mapping, the mapped volume is

$$|(\hat{\mathbf{q}}_3 h_3 dq_3) \cdot ((\hat{\mathbf{q}}_1 h_1 dq_1) \times (\hat{\mathbf{q}}_2 h_2 dy))| = |J_{\mathbf{r}}|\, dq_1 dq_2 dq_3.$$

If $\hat{\mathbf{q}}_\alpha$ are orthonormal, then $|J_{\mathbf{r}}| = h_1 h_2 h_3$.

Note: In $\mathbb{R}^3$ lines and surfaces have directions, but volumes don't. So there are no analogies to line and surface integrals of vector fields. Instead ...

Defn: The *Divergence theorem* or *Guass' Theorem* states that, for a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$,

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F}\, dV = \int_{\partial V} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS \equiv \int_{\partial V} \mathbf{F} \cdot d\mathbf{S}$$

where $d\mathbf{S}$ is a surface element and $\hat{\mathbf{n}}$ points outwards from the volume V.

3.5.2 Outline proof of the divergence theorem

As in Stokes' theorem, start with a proof for a cuboid

$$V = \{\mathbf{r} \,|\, 0 < x < a, 0 < y < b, 0 < z < c\}.$$

The argument will be again that an arbitrary V can divided into many small rectangular volumes over each of which the divergence applies.

We write $\mathbf{F} = F_1\hat{\mathbf{x}} + F_2\hat{\mathbf{y}} + F_3\hat{\mathbf{z}}$. Then it follows that

$$\begin{aligned}\int_{\partial V} \mathbf{v} \cdot d\mathbf{S} \;&=\; \int_0^a \int_0^b (v_3(x, y, c) - v_3(x, y, 0))\, dy\, dx \\ &+ \int_0^a \int_0^c (v_2(x, b, z) - v_2(x, 0, z))\, dz\, dx + \int_0^b \int_0^c (v_1(a, y, z) - v_1(0, y, z))\, dz\, dy.\end{aligned}$$

(there are 6 sides, and the unit outward normal is one of $\pm\hat{\mathbf{x}}$, $\pm\hat{\mathbf{y}}$, $\pm\hat{\mathbf{z}}$ depending on the cuboid side).

Next we consider the volume integral,

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{F}\, dV = \int_0^a \int_0^b \int_0^c \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dz\, dy\, dx.$$

The 3 terms are considered separately but in the same manner. For example, Consider the contribution from $\partial F_3/\partial z$. From the Fundamental Theorem of Calculus,

$$\int_0^a \int_0^b \int_0^c \frac{\partial F_3}{\partial z}\, dz\, dy\, dx = \int_0^a \int_0^b (F_3(x,y,c) - F_3(x,y,0))\, dy\, dx.$$

The result is

$$\begin{aligned}\int_V \boldsymbol{\nabla} \cdot \mathbf{F}\, dV &= \int_0^a \int_0^b (F_3(x,y,c) - F_3(x,y,0))\, dy\, dx \\ &\quad \int_0^a \int_0^c (F_2(x,b,z) - F_2(x,0,z))\, dz\, dx + \int_0^b \int_0^c (F_1(a,y,z) - F_1(0,y,z))\, dz\, dy,\end{aligned}$$

which coincides with (18), thus confirming the theorem.

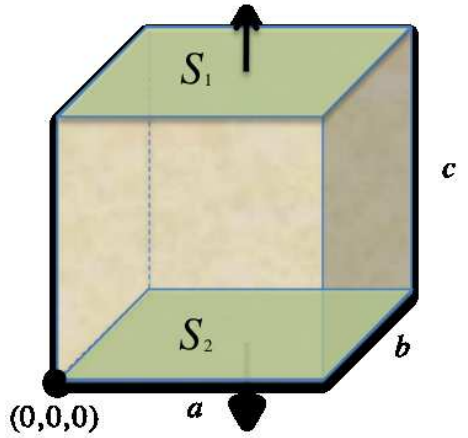

Figure 4: Gauss' theorem for the cuboid V. The top and bottom faces of the boundary, S_1 and S_2, are indicated.

E.g. Let V be the ball of radius a about the origin, and let

$$\mathbf{v}(\mathbf{r}) = \mathbf{r} + f(r)\hat{\mathbf{z}} \times \mathbf{r},$$

where $\hat{\mathbf{z}}$ is the unit vector along the z-axis, and $r = (x^2+y^2+z^2)^{1/2}$. We'll compute the volume integral first. We have that

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 3 + (\boldsymbol{\nabla} f) \cdot (\hat{\mathbf{z}} \times \mathbf{r}).$$

But

$$\boldsymbol{\nabla} f = \frac{f'(r)}{r}\mathbf{r},$$

and $\mathbf{r} \cdot (\hat{\mathbf{z}} \times \mathbf{r}) = 0$, so that

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 3.$$

As the divergence of $\mathbf{v}$ is a constant, its integral over V is just its value times the volume of V,

$$\int_V \boldsymbol{\nabla} \cdot \mathbf{v}\ \mathrm{d}V = 3\frac{4\pi}{3}a^3 = 4\pi a^3.$$

Next, we consider the surface integral. The sphere of radius a is parameterised by

$$\mathbf{r} = \mathbf{s}(\phi, \theta) = a(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) = a\hat{\mathbf{r}}, \quad 0 \le \phi \le \pi, \quad 0 \le \theta < 2\pi.$$

Then

$$\int_{\partial V} \mathbf{v} \cdot d\mathbf{S} = \int_0^{\pi} \int_0^{2\pi} \mathbf{v}(\mathbf{r}(\phi, \theta)) \cdot \mathbf{N}(\phi, \theta)\, d\theta\, d\phi.$$

We have that

$$\frac{\partial \mathbf{r}}{\partial \phi} = a\hat{\boldsymbol{\phi}}, \quad \frac{\partial \mathbf{r}}{\partial \theta} = a\sin\phi\hat{\boldsymbol{\theta}},$$

so that

$$\mathbf{N}(\phi, \theta) = a^2 \sin\phi\hat{\mathbf{r}}.$$

Therefore,

$$\mathbf{v}(\mathbf{r}(\phi, \theta)) \cdot \mathbf{N}(u, v) = (\mathbf{r}(\phi, \theta) + f(r(\phi, \theta))\hat{\mathbf{z}} \times \mathbf{r}(\phi, \theta)) \cdot a^2 \sin\phi\hat{\mathbf{r}} = a^3 \sin\phi.$$

The surface integral is given by

$$\int_{\partial V} \mathbf{v} \cdot d\mathbf{S} = \int_0^{\pi} \int_0^{2\pi} a^3 \sin\phi\, d\theta\, d\phi = 4\pi a^3,$$

and the divergence theorem is verified.

3.5.3 Green's Identities

If $\mathbf{F} = \boldsymbol{\nabla} f$ (i.e. the vector field can be described by a scalar potential) then the divergence theorem reads

$$\int_V \triangle f\, dV = \int_{\partial V} \hat{\mathbf{n}}.\boldsymbol{\nabla} f\, dS$$

If $\mathbf{F} = g\boldsymbol{\nabla} f$, g, f scalar fields then

$$\int_V \boldsymbol{\nabla} g \cdot \boldsymbol{\nabla} f + g\triangle f\, dV = \int_{\partial V} g\hat{\mathbf{n}} \cdot \boldsymbol{\nabla} f\, dS$$

subtracting the result of using $\mathbf{F} = f\boldsymbol{\nabla} g$ we have

$$\int_V (g\triangle f - f\triangle g)\, dV = \int_{\partial V} (g\hat{\mathbf{n}} \cdot \boldsymbol{\nabla} f - f\hat{\mathbf{n}} \cdot \boldsymbol{\nabla} g)\, dS$$

These can be very useful in deriving equations underlying physical applications.