MATH20901 Multivariable Calculus
Problems Class Week 2

$*$ Not covered explicitly in the current course

1. [Exam 2012/13, Q1]

   (a) (i) (3 marks)$*$
   Let $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$. Define what it means that $\mathbf{F}$ is differentiable at a point $\mathbf{x} \in \mathbb{R}^m$.

   (ii) (3 marks)$*$
   Let $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$ be differentiable at $\mathbf{x} \in \mathbb{R}^m$, and let $a \in \mathbb{R}$. Prove that $a\mathbf{F}$ is differentiable at $\mathbf{x}$.

   (iii) (3 marks)
   Give an example of two functions $\mathbf{F}$, $\mathbf{G} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathbf{F} \neq \mathbf{G}$, but $\mathbf{F}'(0) = \mathbf{G}'(0)$.

   (b) (4 marks)
   Consider the transformation $(x, y) = (r\cos\theta, r\sin\theta)$. Show that the Jacobian determinant is given by

   $$\frac{\partial(x,y)}{\partial(r,\theta)} = r.$$

   When can we solve for (r, θ) in terms of (x, y) ?

   (c) (12 marks)
   Show that the pair of equations

   $$x^2 - y^2 - u^3 + v^2 + 4 = 0, \qquad 2xy + y^2 - 2u^2 + 3v^4 + 8 = 0$$

   determine local functions $x(u, v)$ and $y(u, v)$ defined for (u, v) near $u = 2$ and $v = 1$ such that $x(2, 1) = 2$, $y(2, 1) = -1$. Compute $\dfrac{\partial u}{\partial x}$ at $(2, 1)$.

2. [Exam 2013/14, Q1]

   (a) i. (3 marks) Let $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$. Define what it means that $\mathbf{F}$ is differentiable at a point $\mathbf{x} \in \mathbb{R}^m$.

   ii. (5 marks) Let

   $$\mathbf{f}(u, v, w) = (v^2 + uw, u^2 + w^2, u^2 v - w^3),$$

   and

   $$\mathbf{g}(x, y) = (xy^3, x^2 - y^2, 3x + 5y),$$

   and define $\mathbf{h}(x, y) = (\mathbf{f} \circ \mathbf{g})(x, y)$. Compute $\mathbf{h}'(-1, 1)$.

(b) (4 marks) Given

$$z = f\left(\frac{x+y}{x-y}\right),$$

where $f \in C^1(\mathbb{R}^2, \mathbb{R})$, show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

(c) $*$ Define the function $f\colon \mathbb{R}^2 \to \mathbb{R}$ by

$$f(\mathbf{x}) = \begin{cases} \frac{x_1 x_2}{\sqrt{x_1^2+x_2^2}}, & \mathbf{x} \neq 0, \\ 0, & \mathbf{x} = 0. \end{cases}$$

i. (5 marks) Show that all partial derivatives exist for $\mathbf{x} \in \mathbb{R}^2$, and calculate them.

ii. (3 marks) Show that the partial derivatives of f are not continuous at the origin.

iii. (5 marks) Show that f is not differentiable at the origin.