MATH20901 Multivariable Calculus
Problems Class Week 4

1. [Exam 2012/13, Q2]

   (a) (4 marks) Let $\mathbf{v}(x, y, z)$ be the vector field on $\mathbb{R}^3$ given by

   $$\mathbf{v}(x, y, z) = (x \sin z, yz, \cos z).$$

   Compute $\boldsymbol{\nabla} \cdot \mathbf{v}$ and $\boldsymbol{\nabla} \times \mathbf{v}$.

   (b) Cylindrical coordinates are defined by

   $$\mathbf{r}(r, \theta, z) = (r \cos\theta, r \sin\theta, z).$$

   (i) (3 marks) Calculate the basis vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, $\hat{\mathbf{z}}$.

   (ii) (1 marks) Calculate the scale factors h_r, h_θ, and h_z.

   (iii) (7 marks) Calculate $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}$ in cylindrical coordinates, where $\mathbf{u} = u_r\hat{\mathbf{r}} + u_\theta\hat{\boldsymbol{\theta}} + u_z\hat{\mathbf{z}}$.

   (c) (i) (6 marks) Let $f(r)$ be a smooth scalar-valued function of $r = |\mathbf{r}|$, and let $\mathbf{a} \in \mathbb{R}^3$ be a constant vector. Calculate

   $$\boldsymbol{\nabla} \times (\mathbf{r} \times \mathbf{a} f(r))$$

   (ii) (4 marks) Let $\mathbf{u}$ be a vector field in $C(\mathbb{R}^3, \mathbb{R}^3)$. Show that

   $$\mathbf{u} \times (\boldsymbol{\nabla} \times \mathbf{u}) = \frac{1}{2}\boldsymbol{\nabla}(\mathbf{u} \cdot \mathbf{u}) - (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u}$$