MATH20901 Multivariable Calculus
Problems Class Week 6

1. [Exam 2012/13, Q3(a,b,c)]

   (a) (4 marks)
   Let $\mathbf{v}(x,y) = (y^2, -xy)$ be a vector field on $\mathbb{R}^2$, and let C be the part of the circle $x^2+y^2 = 1$ that starts at $(1,0)$ and ends at $(0,1)$, oriented clockwise. Compute $\int_C \mathbf{v} \cdot d\mathbf{r}$.

   (b) (4 marks)
   Let S be the surface of the unit sphere, and let $\mathbf{F} = F_r\hat{\mathbf{r}}$ be a vector field where $\hat{\mathbf{r}} = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ is the unit vector pointing radially outward from the sphere, where $0 \leq \phi \leq \pi$, and $0 \leq \theta \leq 2\pi$. Show that

   $$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi} F_r \sin\phi d\phi d\theta.$$

   (c) Let S be the surface in $\mathbb{R}^3$ given by the cone $z = (x^2+y^2)^{1/2}$, $x^2+y^2 \leq 1$. Let $\mathbf{v}(x,y,z)$ be the vector field given by

   $$\mathbf{v}(x,y,z) = (z^2, xz, y^2)$$

   (i) (8 marks) Without using Stokes theorem, compute $\displaystyle\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{S}$ where $d\mathbf{S}$ is oriented inward towards the z-axis.

   (ii) (5 marks) Recalculate the surface integral from (i) by computing the line integral along an appropriate path.

2. [Exam 2013/14 Q3(a,b)]

   (a) (5 marks) Let $f(\mathbf{r})$ be a scalar field and let C be a curve in $\mathbb{R}^3$ parameterised by the path $\mathbf{p}(s)$, $a \leq s \leq b$. Prove that

   $$\int_C \boldsymbol{\nabla} f \cdot d\mathbf{r} = f(\mathbf{p}(b)) - f(\mathbf{p}(a)).$$

   (b) Consider a sphere of radius R about the origin, parameterized by

   $$\mathbf{s}(\phi,\theta) = R(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi),$$

   with the vector field $\mathbf{f} = (y, -x, z)$.

   (i) (4 marks) Calculate the line integral

   $$\int_C \mathbf{f} \cdot d\mathbf{r},$$

   where C is the closed path along the equator going west (by convention, $\phi = 0$ is the north pole).

   (ii) (5 marks) Verify Stokes' theorem by integrating a suitable vector field over the northern hemisphere.

   (iii) (1 mark) What would the orientation of the surface be if the integration in (ii) had been over the southern hemisphere?