MATH20901 Multivariable Calculus: Solutions 2 [1]

1. (a) $\delta_{ij}\delta_{ij} = \delta_{jj} = n$ using the rule $\delta_{ij}c_i = c_j$.

   (b) $(AB^TC)_{ij} = A_{il}B_{kl}C_{kj} \equiv \sum_{l=1}^{p}\sum_{k=1}^{q} A_{il}B^T_{lk}C_{kj}$ and $B^T_{lk} = B_{kl}$, observing that A has p columns, and C has q rows.

2. A generalisation of the notes to a non-normal basis. We start with $\mathbf{x} = c_j\mathbf{e}_j$. Now, take the inner product with $\mathbf{e}_k$ and we have $\mathbf{x}\cdot\mathbf{e}_k = c_j\mathbf{e}_j\cdot\mathbf{e}_k = c_j\delta_{jk}|\mathbf{e}_j|^2$. And so

$$c_k = \frac{\mathbf{x}\cdot\mathbf{e}_k}{|\mathbf{e}_j|^2}, \qquad \text{whence} \qquad \mathbf{x} = \frac{(\mathbf{x}\cdot\mathbf{e}_k)\mathbf{e}_k}{|\mathbf{e}_j|^2}$$

   This concept can be extended to infinite dimensional space, where vectors become functions. Thus arbitrary functions can be expanded in exactly the same way in terms of sets of basis functions – a branch of mathematics called 'functional analysis' briefly touched on in the APDE2 course.

   If $\mathbf{e}_i$ are non-orthogonal then you can still proceed as before to get $\mathbf{x}\cdot\mathbf{e}_k = c_j\mathbf{e}_j\cdot\mathbf{e}_k$ but you can't introduce the Kronecker delta any more. But you can still write down a matrix system for the unknowns, c_j:

$$\begin{pmatrix} \mathbf{e}_1\cdot\mathbf{e}_1 & \mathbf{e}_2\cdot\mathbf{e}_1 & \dots & \\ \mathbf{e}_1\cdot\mathbf{e}_2 & \mathbf{e}_2\cdot\mathbf{e}_2 & \dots & \\ \vdots & & \ddots & \\ & & & \mathbf{e}_m\cdot\mathbf{e}_m \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} \mathbf{x}\cdot\mathbf{e}_1 \\ \mathbf{x}\cdot\mathbf{e}_2 \\ \vdots \\ \mathbf{x}\cdot\mathbf{e}_m \end{pmatrix}$$

   which you can invert (in principle) to find the c_j.

3. (a) $\boldsymbol{\nabla} f(\mathbf{r}) = (-y\sin(xy), -x\sin(xy) - z\sin(yz), -y\sin(yz))$. So

$$\begin{aligned} \boldsymbol{\nabla}\times\boldsymbol{\nabla} f &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ -y\sin(xy) & -x\sin(xy) - z\sin(yz) & -y\sin(yz) \end{vmatrix} \\ &= (-\sin(yz) - yz\cos(yz) + \sin(yz) + yz\cos(yz))\hat{\mathbf{x}} + \ldots = 0 \end{aligned}$$

   Has to be so, as proved in notes for any f.

   (b) $\boldsymbol{\nabla}\cdot\mathbf{u} = \sin z + z - \sin z = z$.

   (c)

$$\boldsymbol{\nabla}\times\mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ ayz & bzx & cxy \end{vmatrix} = ((c-b)x, (a-c)y, (b-a)z).$$

   Then $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\mathbf{v}) = (c-b) + (a-c) + (b-a) = 0$ as required from the proof in the notes.

4. (a) Here $f = \mathbf{a} \cdot \mathbf{r}$ and $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{r} = (x_1, x_2, x_3)$ so $f = a_j x_j$ and $[\boldsymbol{\nabla} f]_i = \dfrac{\partial}{\partial x_i}(a_j x_j) = a_j \delta_{ij} = a_i$; thus $\boldsymbol{\nabla} f(\mathbf{r}) = \mathbf{a}$.

(b) First, $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$ so $\boldsymbol{\nabla} r = (x, y, z)/r = \mathbf{r}/r$. Next

$$\mathbf{v} = \boldsymbol{\nabla} r^n = n r^{n-1} \boldsymbol{\nabla} r = n r^{n-2} \mathbf{r}$$

Continuing, we have

$$\boldsymbol{\nabla} \cdot \mathbf{v} = \frac{\partial v_i}{\partial x_i} = \frac{\partial}{\partial x_i}(n x_i r^{n-2}) = n \frac{\partial x_i}{\partial x_i} r^{n-2} + n x_i \left((n-2) x_i r^{n-4} \right) = 3 n r^{n-2} + n(n-2) r^2 r^{n-4}$$

after using the first differentiation result again for the second term of the product. So

$$\boldsymbol{\nabla} \cdot \mathbf{v} = n(n+1) r^{n-2}$$

which vanishes for $n = 0$ and $n = -1$, provided $r \neq 0$. We had to expect that $n = 0$ was one solution as $r^0 = 1$.

(c) Here $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ and $\mathbf{r} = (x_1, x_2, x_3)$. The ith component of the curl is

$$\begin{aligned}[\boldsymbol{\nabla} \times \mathbf{v}]_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j} v_k = \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{klm} \omega_l x_m = \epsilon_{ijk} \epsilon_{klm} \omega_l \frac{\partial x_m}{\partial x_j} = \epsilon_{ijk} \epsilon_{klm} \omega_l \delta_{jm} = \epsilon_{imk} \epsilon_{klm} \omega_l \\ &= \epsilon_{kim} \epsilon_{klm} \omega_l = (\delta_{il} \delta_{mm} - \delta_{im} \delta_{ml}) \omega_l = (3\delta_{il} - \delta_{il}) \omega_l = 2\omega_i.\end{aligned}$$

So we have $\boldsymbol{\nabla} \times \mathbf{v} = 2\boldsymbol{\omega}$.

5. (a)(i) Similar to 4(c) above, but we also have $\boldsymbol{\nabla} r = \mathbf{r}/r$ or

$$\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$$

The i component is

$$\begin{aligned}[\boldsymbol{\nabla} \times (\mathbf{r} \times \mathbf{a} f(r))]_i &= \epsilon_{ijk} \frac{\partial}{\partial x_j} \epsilon_{klm} x_l a_m f(r) = \epsilon_{kij} \epsilon_{klm} \left(\frac{\partial x_l}{\partial x_j} a_m f(r) + x_l a_m \frac{x_j}{r} f'(r) \right) \\ &= \epsilon_{kij} \epsilon_{klm} \left(\delta_{lj} a_m f(r) + a_m \frac{x_l x_j}{r} f'(r) \right) \\ &= (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \left(\delta_{lj} a_m f(r) + a_m \frac{x_l x_j}{r} f'(r) \right) \\ &= a_i f(r) - \delta_{jj} a_i f(r) + a_j \frac{x_i x_j}{r} f'(r) - a_i \frac{x_j^2}{r} f' \\ &= \left[-\mathbf{a}(2f(r) + r f'(r)) + \mathbf{r} \frac{\mathbf{a} \cdot \mathbf{r}}{r} f'(r) \right]_i\end{aligned}$$

and so

$$\boldsymbol{\nabla} \times (\mathbf{r} \times \mathbf{a} f(r)) = -\mathbf{a}(2f(r) + r f'(r)) + \mathbf{r} \frac{\mathbf{a} \cdot \mathbf{r}}{r} f'(r).$$

(a)(ii) Same tricks as above, slightly easier now

$$[\boldsymbol{\nabla} \cdot \mathbf{a} f(r)]_i = \frac{\partial}{\partial x_i}(a_i f(r)) = a_i \frac{\partial f}{\partial x_i} = a_i \frac{x_i}{r} f'(r)$$

using (a)(i). So $\nabla \cdot \mathbf{a} f(r) = \dfrac{\mathbf{a}\cdot\mathbf{r}}{r} f'(r)$.

(b)

$$\begin{aligned}
[\mathbf{u}\times(\nabla\times\mathbf{u})]_i &= \epsilon_{ijk} u_j \epsilon_{klm} \frac{\partial}{\partial x_l} u_m = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, u_j \frac{\partial}{\partial x_l} u_m \\
&= u_m \frac{\partial}{\partial x_i} u_m - u_l \frac{\partial}{\partial x_l} u_i = \frac{\partial}{\partial x_i}\left(\tfrac{1}{2}u_m^2\right) - u_l \frac{\partial}{\partial x_l} u_i = \left(\tfrac{1}{2}\nabla \mathbf{u}^2 - [\mathbf{u}\cdot\nabla)\mathbf{u}\right]_i,
\end{aligned}$$

which is the ith component of $\mathbf{u}\times(\nabla\times\mathbf{u}) = \frac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - (\mathbf{u}\cdot\nabla)\mathbf{u}$.

6. (a) $\nabla\cdot(f\mathbf{v}) = \dfrac{\partial}{\partial x_i}(f v_i) = f\dfrac{\partial v_i}{\partial x_i} + v_i \dfrac{\partial f}{\partial x_i} = f\nabla\cdot\mathbf{v} + \mathbf{v}\cdot\nabla f$

   (b) This is just part (a) with $\mathbf{v} = \nabla g$ and the only thing to note here is that $\nabla\cdot\nabla g = \Delta g$, the Laplacian of g.

   (c) Take the ith component of the LHS:

$$[\nabla\times(f\mathbf{v})]_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}(f v_k) = \epsilon_{ijk} f \frac{\partial v_k}{\partial x_j} + \epsilon_{ijk}\frac{\partial f}{\partial x_j} v_k = f[\nabla\times\mathbf{v}]_i + [\nabla f\times\mathbf{v}]_i$$

7. On the LHS if you switch over $\mathbf{u}$ and $\mathbf{v}$, by the definition of the cross product you will introduce a minus sign. However the RHS is symmetric in $\mathbf{u}$ and $\mathbf{v}$ and so switching them over will give the same result. So it cannot be true as stated. Here's the derivation.

$$\begin{aligned}
\nabla\cdot(\mathbf{u}\times\mathbf{v}) &= \frac{\partial}{\partial x_i}(\epsilon_{ijk} u_j v_k) = \epsilon_{ijk}\left(u_j \frac{\partial v_k}{\partial x_i} + v_k \frac{\partial u_j}{\partial x_i}\right) \\
&= -u_j \epsilon_{jik} \frac{\partial v_k}{\partial x_i} + v_k \epsilon_{kij} \frac{\partial u_j}{\partial x_i} \\
&= -\mathbf{u}\cdot(\nabla\times\mathbf{v}) + \mathbf{v}\cdot(\nabla\times\mathbf{u})
\end{aligned}$$

   In the above we have used the cyclic definition of ϵ_{ijk}.

8. Similar to above

$$\begin{aligned}
[\nabla\times(\mathbf{u}\times\mathbf{v})]_i &= \epsilon_{ijk}\frac{\partial}{\partial x_j}(\epsilon_{klm} u_l v_m) = \epsilon_{kij}\epsilon_{klm}\left(u_l \frac{\partial v_m}{\partial x_j} + v_m \frac{\partial u_l}{\partial x_j}\right) \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\left(u_l \frac{\partial v_m}{\partial x_j} + v_m \frac{\partial u_l}{\partial x_j}\right) = u_i \frac{\partial v_j}{\partial x_j} + v_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial v_i}{\partial x_j} - v_i \frac{\partial u_j}{\partial x_j} \\
&= (\nabla\cdot\mathbf{v}) u_i + (\mathbf{v}\cdot\nabla) u_i - (\mathbf{u}\cdot\nabla) v_i - (\nabla\cdot\mathbf{u}) v_i
\end{aligned}$$

   So we match up each suffix to give the vector result

9. (a) Here $\mathbf{F} = -\rho g \hat{\mathbf{z}} = \nabla\phi$. So

$$\phi_x = 0, \quad \phi_y = 0, \quad \phi_z = -\rho g$$

   Integrating each up gives $\phi(x,y,z) = f_1(y,z)$, $\phi(x,y,z) = f_2(x,z)$ and $\phi(x,y,z) = -\rho g z + f_3(x,y)$ where f_1, f_2, f_3 are arbitrary functions. In fact, the only way this can resolve itself is if $f_1 = f_2 = f_3 = C$ a constant. So, in general $\phi = -\rho g z + C$.

(b) You do this in Fluids 3. We have

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} = -\frac{1}{\rho}\boldsymbol{\nabla}p + \boldsymbol{\nabla}\phi$$

after using part (a). Using Q3(b) we have

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}\boldsymbol{\nabla}(\mathbf{u}\cdot\mathbf{u}) - \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u}) = -\frac{1}{\rho}\boldsymbol{\nabla}p + \boldsymbol{\nabla}\phi$$

Taking the curl, noting that $\boldsymbol{\nabla}\times\boldsymbol{\nabla}f = 0$ for any scalar f, and defining $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\mathbf{u}$ we have

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{\nabla}\times(\mathbf{u}\times\boldsymbol{\omega}) = 0$$

Now using Q6, we can write

$$\frac{\partial \boldsymbol{\omega}}{\partial t} - (\boldsymbol{\nabla}\cdot\boldsymbol{\omega})\mathbf{u} + (\boldsymbol{\nabla}\cdot\mathbf{u})\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} = 0$$

Finally, since $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$ and $\boldsymbol{\nabla}\cdot\boldsymbol{\omega} = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}\times\mathbf{u} = 0$, we can write

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{u}$$

(c) If $\mathbf{u} = (u_1(x,y), u_2(x,y), 0)$, then $\boldsymbol{\omega} = (\partial_x u_2 - \partial_y u_1)\hat{\mathbf{z}}$ and both $(\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{u} = 0$ and $(\mathbf{u}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} = 0$. Which means that $\partial_t\boldsymbol{\omega} = 0$ and so $\boldsymbol{\omega}$ is constant.

10. Using result (i) in the question we can write Navier's equation as

$$\rho\frac{\partial^2 \mathbf{u}}{\partial t^2} = (\lambda + 2\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{u}) - \mu\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{u})$$

Now taking the divergence of this and using result (iii) on the last term to eliminate it gives

$$\rho\frac{\partial^2}{\partial t^2}(\boldsymbol{\nabla}\cdot\mathbf{u}) = (\lambda + 2\mu)\Delta(\boldsymbol{\nabla}\cdot\mathbf{u})$$

and letting $\phi = \boldsymbol{\nabla}\cdot\mathbf{u}$ and $c_1^2 = (\lambda + 2\mu)/\rho$ we have the set equation.

Now take the curl and let $\mathbf{H} = \boldsymbol{\nabla}\times\mathbf{u}$ we can eliminate the first term on the RHS to leave

$$\rho\frac{\partial^2}{\partial t^2}\mathbf{H} = \mu\boldsymbol{\nabla}\times\Delta\mathbf{u} = -\mu\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{H})$$

once results (i), (ii) are used. We need one more result, given in §2.3.1 of the notes

$$\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\mathbf{H}) = \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{H}) - \Delta\mathbf{H})$$

and since $\mathbf{H} = \boldsymbol{\nabla}\times\mathbf{u}$ the first term is zero. Hence we have

$$\rho\frac{\partial^2}{\partial t^2}\mathbf{H} = \mu\Delta\mathbf{H}$$

as required with $c_2^2 = \mu/\rho$.

The reduction of Navier's equation to these two decoupled equations is very important in the study of Seismology as they represent *wave equations* for the dilation (or compressible) and rotational components of displacements in a solid. The factors c_1 and c_2 are *wave speeds* (see APDE2) and clearly $c_1 > c_2$. This means compression waves travel faster than rotational waves. In Siesmology the two waves and called P and S waves – the P is for primary (because they arrive first) and the S for secondary.