MATH20901 Multivariable Calculus: Solutions 4 [1]

1. We have that
$$L = \int_C |d\mathbf{r}| = \int_0^{2\pi} |\mathbf{p}'(t)| dt$$

   under parametrisation and $\mathbf{p}'(t) = a(1 - \cos t, \sin t, 0)$ so

   $$L = a \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt = a \int_0^{2\pi} \sqrt{4\sin^2(t/2)}\, dt = 4a\left[-\cos(t/2)\right]_0^{2\pi} = 8a$$

   So the nail travels exactly 4 diameters of the wheel. If the wheel were not moving along the ground, i.e. only rotating, the nail would travel π diameters (the circumference of the wheel). So it actually doesn't go much further on account of its translation.

2. The curve C is a helix with an axis coinciding with the z-axis. We have that

   $$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_0^1 \mathbf{v}(\mathbf{p}(t)) \cdot \mathbf{p}'(t)\, dt.$$

   where $\mathbf{p}(t)$ is the path along the curve and

   $$\mathbf{p}'(t) = \left(\frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right), -\frac{\pi}{2}\sin\left(\frac{\pi}{2}t\right), 1\right),$$

   whilst
   $$\mathbf{v}(\mathbf{p}(t)) = \left(\sin\left(\frac{\pi}{2}t\right), \sin\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}t\right), \sin\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}t\right)t\right).$$

   Substituting, we get

   $$\int_C \mathbf{v} \cdot d\mathbf{r} = \int_0^1 \left(\frac{\pi}{2}\sin\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}t\right) - \frac{\pi}{2}\sin^2\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}t\right) + \sin\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}t\right)t\right) dt.$$

   The first two terms in the integrand above may be integrated easily (longhand by substitution $u = \sin(\pi t/2)$ if you need to) to obtain

   $$\left[\frac{1}{2}\sin^2\left(\frac{\pi}{2}t\right) - \frac{1}{3}\sin^3\left(\frac{\pi}{2}t\right)\right]_0^1 = \frac{1}{6}.$$

   The last term needs integration by parts:

   $$\int_0^1 \sin\left(\frac{\pi}{2}t\right)\cos\left(\frac{\pi}{2}t\right)t\, dt = \frac{1}{2}\int_0^1 \sin(\pi t)t\, dt = \left[-\frac{t}{2\pi}\cos(\pi t)\right]_0^1 + \frac{1}{2\pi}\int_0^1 \cos(\pi t)\, dt = \frac{1}{2\pi}.$$

   Therefore,
   $$\int_C \mathbf{v} \cdot d\mathbf{r} = \frac{1}{6} + \frac{1}{2\pi}.$$

---

3. Here $dS = |\hat{n}dS|$ and we have $(x, y) = \mathbf{s}(r, \theta) = (ra\cos\theta, rb\sin\theta)$ and $D = \{(r, \theta)\,|\,0 < r <$
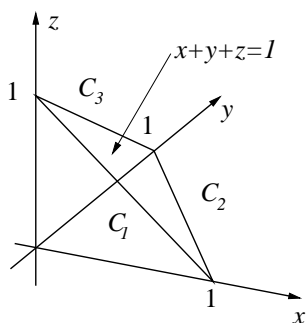1, $0 < \theta < 2\pi\}$ which means

$$\hat{\mathbf{n}}dS = \frac{\partial\mathbf{s}}{\partial r} \times \frac{\partial\mathbf{s}}{\partial\theta}dr\,d\theta = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ a\cos\theta & b\sin\theta & 0 \\ -ra\sin\theta & rb\cos\theta & 0 \end{vmatrix}dr\,d\theta = rab\,dr\,d\theta\hat{\mathbf{z}}$$

So the area of the ellipse is $\int_S dS = \int_D |rab\hat{\mathbf{z}}|\,dr\,d\theta = 2\pi ab\int_0^1 r\,dr = \pi ab$ In this map of a 2D
surface to a 2D surface, we not that the factor rab is just the Jacobian determinant.

4. There are 4 segments to the square: (i) on the path from $(0,0)$ to $(0,l)$, $y = 0$ and $d\mathbf{r} = dx\hat{\mathbf{x}}$;
(ii) on the path from $(0,l)$ to (l,l) $x = l$ and $d\mathbf{r} = dy\hat{\mathbf{y}}$; (iii) on the path from (l,l) to $(l,0)$,
$y = l$ and $d\mathbf{r} = dx\hat{\mathbf{x}}$; (iv) on the path from $(l,0)$ to $(0,0)$, $x = 0$ and $d\mathbf{r} = dy\hat{\mathbf{y}}$. So we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^l (0,0,0)\cdot\hat{\mathbf{x}}dx + \int_0^l (-l^2y, ly^2, 0)\cdot\hat{\mathbf{y}}dy + \int_l^0 (-x^2l, xl^2, 0)\cdot\hat{\mathbf{x}}dx + \int_l^0 (0,0,0)\cdot\hat{\mathbf{y}}dy$$
$$= 0 + l\int_0^l y^2dy + l\int_0^l x^2dx + 0 = \frac{2l^4}{3}$$

5. (a) See figure. The plane $x + y + z = 1$ intersects with the plane $y = 0$ along the straightline
segment $C_1 = \{y = 0, z = 1 - x\}$, with the plane $z = 0$ along $C_2 = \{z = 0,\ y = 1 - x\}$ and
with the plane $x = 0$ along $C_3 = \{x = 0, z = 1 - y\}$.

(b) Need to parametrise the curve C. Do each line segment individually and make sure each
segment is oriented in the same sense. So, C_1, C_2, C_3 are described (respectively) by the three
paths
$$\mathbf{p}_1(t) = (t, 0, 1 - t), \quad \mathbf{p}_2(t) = (1 - t, t, 0), \quad \mathbf{p}_3(t) = (0, 1 - t, t)$$
each holding for $0 < t < 1$. So
$$\mathbf{p}_1'(t) = (1, 0, -1), \quad \mathbf{p}_2'(t) = (-1, 1, 0), \quad \mathbf{p}_3'(t) = (0, -1, 1)$$

First,
$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (t^2(1 - t), 0, (1 - t)^2)\cdot(1, 0, -1)\,dt = \int_0^1 (-t^3 - 1 + 2t)\,dt = -\frac{1}{4}$$

Next,
$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (0, t^2(1 - t), 0)\cdot(-1, 1, 0)\,dt = \int_0^1 (t^2 - t^3)\,dt = \tfrac{1}{3} - \tfrac{1}{4} = \frac{1}{12}$$

2

Finally,

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (0, 0, t^2) \cdot (0, -1, 1)\, dt = \int_0^1 t^2\, dt = \frac{1}{3}$$

Since the curves are all oriented clockwise, we sum over each contribution to give

$$\int_C \mathbf{F} \cdot d\mathbf{r} = -\frac{1}{4} + \frac{1}{12} + \frac{1}{3} = \frac{1}{6}$$

(c) Now the surface integral. Any surface with edges coinciding with the closed curve C will do. Make sense to use the plane $x + y + z = 1$. We want to parametrise the curve so we use

$$D = \{(u, v) \,|\, 0 < v < 1 - u, \ 0 < u < 1\}$$

and write $(x, y, z) = \mathbf{s}(u, v) = (u, v, 1-u-v)$ (this is just the projection of the slanted triangular section onto the x, y-plane). So

$$\mathbf{N} = \frac{\partial \mathbf{s}}{\partial u} \times \frac{\partial \mathbf{s}}{\partial v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \hat{\mathbf{x}} + \hat{\mathbf{y}} + \hat{\mathbf{z}}$$

Next,

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ x^2 z & xy^2 & z^2 \end{vmatrix} = x^2 \hat{\mathbf{y}} + y^2 \hat{\mathbf{z}}$$

Then

$$\begin{aligned}
\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} &= \int_D (u^2 \hat{\mathbf{y}} + v^2 \hat{\mathbf{z}}) \cdot (1, 1, 1)\, du dv = \int_0^1 \int_0^{1-u} u^2 + v^2\, dv\, du \\
&= \int_0^1 \left[u^2 v + \frac{v^3}{3} \right]_0^{1-u} du = u^2(1-u) + \frac{(1-u)^3}{3}\, du = \left[\frac{u^3}{3} - \frac{u^4}{4} - \frac{(1-u)^4}{12} \right]_0^1 \\
&= \frac{1}{3} - \frac{1}{4} + \frac{1}{12} = \frac{1}{6}
\end{aligned}$$

The same as part (b) by Stokes' theorem.

(d) If $\mathbf{F} = (yz, xz, xy)$ then we see that $\mathbf{F} = \nabla(xyz)$ and hence $\nabla \times \nabla(xyz) = 0$ by an identity. Hence the integral calculated in (c) is zero and (b) is zero also by Stokes' theorem.

6. (a) Since the curve C can be projected onto the unit circle in the (x, y)-plane we parametrised by writing

$$\mathbf{p}(t) = (\cos t, \sin t, 2 - \sin t), \qquad 0 < t < 2\pi$$

Then $\mathbf{F}(\mathbf{p}(t)) = (-\sin^2 t, \cos t, (2 - \sin t)^2)$, whilst $\mathbf{p}'(t) = (-\sin t, \cos t, -\cos t)$ and so

$$\begin{aligned}
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (-\sin^2 t, \cos t, (2 - \sin t)^2) \cdot (-\sin t, \cos t, -\cos t)\, dt \\
&= \int_0^{2\pi} \left(-\sin^3 t + \cos^2 t - \cos t (2 - \sin t)^2 \right) dt \\
&= \int_0^{2\pi} \left(-\sin t (1 - \cos^2 t) + \frac{1}{2} + \frac{1}{2}\cos 2t - \cos t (2 - \sin t)^2 \right) dt \\
&= \left[\cos t - \frac{\cos^3 t}{3} + \frac{t}{2} + \frac{\sin 2t}{4} + \frac{(2 - \sin t)^3}{3} \right]_0^{2\pi} = \pi
\end{aligned}$$

3

(could have spotted that only the $\cos^2 t$ counts to this integral and made the calculation shorter.)

(b) We project the surface onto the unit circle in to (x, y)-plane so define $D = \{(r, \theta) \mid 0 < r < 1, \; 0 < \theta < 2\pi\}$ and define the surface S with

$$\mathbf{s}(r, \theta) = (r \cos\theta, r \sin\theta, (2 - r \sin\theta))$$

Then

$$\mathbf{N} = \frac{\partial \mathbf{s}}{\partial r} \times \frac{\partial \mathbf{s}}{\partial \theta} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \cos\theta & \sin\theta & -\sin\theta \\ -r \sin\theta & r \cos\theta & -r \cos\theta \end{vmatrix} = r\hat{\mathbf{y}} + r\hat{\mathbf{z}}$$

Easy to show (follow answer to Q5(c)) that $\boldsymbol{\nabla} \times \mathbf{F} = (1 - 2y)\hat{\mathbf{z}} = (1 - 2r \sin\theta)\hat{\mathbf{z}}$. So

$$\int_S \boldsymbol{\nabla} \times \mathbf{F} \cdot d\mathbf{S} = \int_D r(1 - 2r \sin\theta) \, dr \, d\theta = \int_0^{2\pi} \int_0^1 (r - 2r^2 \sin\theta) \, dr \, d\theta = \pi + \frac{2}{3} \left[\cos\theta\right]_0^{2\pi} = \pi$$

Same as (a) by Stokes' theorem.

7. Given D we have

$$\begin{aligned} \int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \, dy &= \int_c^d \int_a^b \frac{\partial g}{\partial x} \, dx \, dy - \int_a^b \int_c^d \frac{\partial f}{\partial y} \, dy \, dx \\ &= \int_c^d [g(x, y)]_{x=a}^{x=b} \, dy - \int_a^b [f(x, y)]_{y=c}^{y=d} \, dx \\ &= \int_c^d g(b, y) \, dy + \int_d^c g(a, y) \, dy + \int_a^b f(x, c) \, dx + \int_b^a f(x, d) \, dx \end{aligned}$$

after reversing the limits to absorb minus signs. We see that the four integrals circumnavigate the edge of the rectangle in an anticlockwise sense.

In other words

$$\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) (x, y) \, dx \, dy = \int_{\partial D} (f, g) \cdot d\mathbf{r} = \int_{\partial D} f \, dx + g \, dy.$$

8. Two ways of doing this: (i) We note that $f\boldsymbol{\nabla}g + g\boldsymbol{\nabla}f = \boldsymbol{\nabla}(fg)$ and since $\boldsymbol{\nabla} \times \boldsymbol{\nabla}(fg) = 0$ regardless of f, g then

$$\int_C \boldsymbol{\nabla}(fg) \cdot d\mathbf{r} = \int_S \boldsymbol{\nabla} \times \boldsymbol{\nabla}(fg) \cdot d\mathbf{S} = 0$$

(ii) We parametrise C by $\mathbf{p}(t)$, $t_1 < t < t_2$ and $\mathbf{p}(t_1) = \mathbf{p}(t_2)$ since C is closed and so

$$\int_C \boldsymbol{\nabla}(fg) \cdot d\mathbf{r} = \int_{t_1}^{t_2} \boldsymbol{\nabla}(fg)(\mathbf{p}(t)) \cdot \mathbf{p}'(t) \, dt = \int_{t_1}^{t_2} \frac{d(fg)}{dt}(\mathbf{p}(t)) \, dt = (fg)(\mathbf{p}(t_2)) - (fg)(\mathbf{p}(t_1)) = 0$$

which is the fundamental theorem of calculus, as in the notes.