UNIVERSITY OF BRISTOL

School of Mathematics

Numerical Analysis 23
MATH 30010
(Paper code MATH–30010)

May/June 2021   2 hours 30 minutes

This paper contains FOUR questions. All answers will be used for assessment.

Calculators of an approved type (non-programmable, no text facility) are allowed in this examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

1. (a) (i) **(5 marks)** Find an LU decomposition of the matrix

$$A = \begin{bmatrix} 1 & 3 & \lambda \\ 2 & 4 & 2 \\ 1 & 4 & 1 \end{bmatrix},$$

where λ is some constant.

(ii) **(5 marks)** Apply your result of part (i) to solve the linear system

$$\begin{bmatrix} 1 & 3 & \lambda \\ 2 & 4 & 2 \\ 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ \mu \end{bmatrix},$$

assuming that it has a unique solution. Here μ is another constant.

(iii) **(5 marks)** For what values of the parameters λ and μ does the linear system in part (ii) have (A) no solution, (B) one unique solution or (C) infinitely many solutions.

(b) **(10 marks)** The following fixed-point iterations are proposed to calculate $7^{1/2}$:

(i)

$$x_{n+1} = \frac{x_n}{2} + \frac{7}{2x_n},$$

(ii)

$$x_{n+1} = \frac{6x_n}{7} + \frac{1}{x_n},$$

(iii)

$$x_{n+1} = \frac{3x_n}{8} + \frac{21}{4x_n} - \frac{49}{8x_n^3}.$$

Determine in each case the order of convergence. Find also the asymptotic error constant in each case.

2. (a) **(7 marks)** Let $P_n(x)$, $n = 0, 1, \ldots$, be a sequence of polynomials of degree n that are orthogonal on the interval $(0, \infty)$ with respect to the weight function $w(x) = x\,\mathrm{e}^{-x}$. Hence they satisfy the orthogonality relation

$$\int_0^\infty x\,\mathrm{e}^{-x}\,P_n(x)\,P_m(x)\,\mathrm{d}x = 0 \qquad \text{if} \qquad n \neq m\,.$$

In addition they satisfy the standardisation condition $P_n(0) = n + 1$ for all n. Determine the first three polynomials $P_0(x)$, $P_1(x)$ and $P_2(x)$.
(Hint: You should find that $P_1(x) = -x + 2$. The following integral may be helpful.)

$$\int_0^\infty x^m\,\mathrm{e}^{-x}\,\mathrm{d}x = m!\,.$$

(b) **(3 marks)** The polynomials $P_n(x)$ satisfy the recursion relation

$$P_n(x) = \left(2 - \frac{x}{n}\right) P_{n-1}(x) - P_{n-2}(x).$$

Apply this relation to check your result for $P_2(x)$ from part (a) and determine the next polynomial $P_3(x)$.

(c) **(5 marks)** Find the sampling points $\{x_1, x_2\}$ and the weights $\{w_1, w_2\}$ for the 2-point Gaussian quadrature formula

$$\int_0^\infty x\,\mathrm{e}^{-x} f(x)\,\mathrm{d}x \approx \sum_{j=1}^{2} w_j\,f(x_j).$$

(d) **(3 marks)** Use the results of part (c) to find an approximation for the integral

$$\int_0^\infty \mathrm{e}^{-x^2}\,\mathrm{d}x.$$

You are not asked to evaluate this approximation numerically.

(e) **(7 marks)** An explicit formula for the polynomials $P_n(x)$ has the form

$$P_n(x) = \sum_{i=0}^{n} \binom{n+1}{n-i} \frac{(-x)^i}{i!}.$$

Apply this formula to show that the polynomial $P_1(x)$ is orthogonal to all other polynomials $P_n(x)$, $n \neq 1$, on the interval $(0, \infty)$ with respect to the weight function $w(x) = x\,\mathrm{e}^{-x}$.

3. (a) Consider the initial value problem

$$y'' - y' - 6y = 0, \quad y(0) = 1, \quad y'(0) = -2. \tag{1}$$

(i) **(4 marks)** Transform this second-order ODE into a system of two first-order ODEs. Apply Euler's method to obtain a system of first-order difference equations. Finally, transform this system of first-order difference equations into one second-order difference equation of the form

$$y_{i+2} + Ay_{i+1} + By_i = 0 \tag{2}$$

where A and B are constants. Specify also the appropriate initial conditions. (Hint: you should find that $A = -2 - h$ and $B = 1 + h - 6h^2$.)

(ii) **(9 marks)** Find the general solution of the difference equation (2) by seeking solutions of the form $y_i = r^i$ where r is a constant to be determined. By imposing the initial conditions, obtain a solution y_i that is an approximation to the solution $y(t_i)$ of the original problem (1), where $t_i = ih$ and h is the step size. Compare the approximate solution y_i to the exact solution $y(t_i)$ and hence show that the error at time t_i, for fixed i, is $O(h^2)$.

(b) (i) **(6 marks)** A linear multistep formula has the form

$$y_{i+1} = \alpha_1 y_i + h\beta_0 f(t_{i+1}, y_{i+1}) + h\beta_2 f(t_{i-1}, y_{i-1}).$$

Find the coefficients α_1, β_0 and β_2 by requiring that the local truncation error is as small as possible. What is the resulting order of accuracy of the formula?

(ii) **(6 marks)** The backward differentiation formula BD3 is given by

$$y_{i+1} = \frac{18}{11}y_i - \frac{9}{11}y_{i-1} + \frac{2}{11}y_{i-2} + \frac{6}{11}hf(t_{i+1}, y_{i+1}).$$

Determine whether this formula is stable quoting any theorem you use. (Hint: one root of the characteristic polynomial is $z = 1$.)

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4. A linear boundary value problem is defined by

$$y''(x) - 3y'(x) + 2y(x) = 2x\,, \quad y(0) = \frac{3}{2}\,, \quad y(1) = \frac{5}{2}\,. \tag{3}$$

(a) **(10 marks)** The boundary value problem (3) may be solved numerically by using the finite difference approach. Apply the central difference approximations for $y'(x)$ and $y''(x)$ and show how the problem reduces to a linear system $A\mathbf{y} = \mathbf{b}$ where the matrix A and the vectors $\mathbf{y}$ and $\mathbf{b}$ should be explicitly given.

(b) **(3 marks)** Consider the linear boundary value problem

$$y''(x) = p(x)\,y'(x) + q(x)\,y(x) + r(x), \quad y(a) = \alpha, \quad y(b) = \beta.$$

Describe how the linear shooting method solves this boundary value problem by transforming it into two initial value problems.

(c) **(7 marks)** Apply the linear shooting method of part (b) to the boundary value problem (3). Solve the corresponding initial value problems exactly and hence obtain a solution to the linear boundary value problem (3).

(d) **(5 marks)** In analogy to the linear shooting method in part (b), describe how you would apply a linear shooting method to a boundary value problem for a third-order equation of the form

$$y'''(x) = p(x)\,y''(x) + q(x)\,y'(x) + r(x)\,y(x) + s(x),\ y(a) = \alpha,\ y'(a) = \beta,\ y(b) = \gamma.$$

End of examination.