UNIVERSITY OF BRISTOL

School of Mathematics

**Numerical Analysis 23**
MATH 30010
(Paper code MATH–30010)

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May/June 2022   2 hours 30 minutes

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This paper contains FOUR questions. All answers will be used for assessment.

Calculators of an approved type (non-programmable, no text facility) are allowed in this examination.

**Candidates may bring four sheets of A4 notes written double-sided into the examination.**

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

*Do not turn over until instructed.*

1. (a) (i) (*6 marks*)

Perform an $LU$ decomposition of the matrix

$$A = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

(all arithmetic should be done exactly and expressed as fractions rather than decimals).

(ii) (*3 marks*)

Show that the matrix above may also be expressed in the form $A = LDL^T$ where $D$ is a diagonal matrix. Hence, or otherwise, show that determinant of $A$ equals $1/2160$.

(iii) (*3 marks*)

Solve the system of equations $\mathbf{A}\mathbf{x} = \mathbf{b}$ for $\mathbf{x}$ when $\mathbf{b} = (1, 1, 1)^T$.

(b) (i) (*4 marks*)

Consider now calculating the solution to the same system as in (a)(iii), but using finite precision arithmetic with three significant-figure accuracy. Thus, the matrix and vector are stored as

$$A = \begin{bmatrix} 1.00 & 0.500 & 0.333 \\ 0.500 & 0.333 & 0.250 \\ 0.333 & 0.250 & 0.200 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \end{bmatrix}.$$

Use Gaussian elimination to determine the numerical approximation to $\mathbf{x}$.

(ii) (*4 marks*)

You should find that there is a significant difference between the answers to (a)(iii) and (b)(i). Provide a plausible explanation for this. Discuss how strategies that might be used to improve the accuracy of the numerical approximation would apply in this case.

(c) (*5 marks*)

Determine the inverse of the $n \times n$ banded matrix

$$\begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}.$$

*Continued...*

2. (a) Consider the nonlinear equation

$$x = 1 - \frac{x^2}{4}. \tag{1}$$

(i) (*5 marks*)
Apply the fixed-point theorem to show that the fixed-point iteration

$$x_{n+1} = 1 - \frac{x_n^2}{4}, \qquad n = 0, 1, 2, \ldots$$

with $x_0 \in [0, 1]$ converges to a unique solution $x^* \in [0, 1]$ of (1).

(ii) (*3 marks*)
Determine the order of convergence of the fixed-point iteration in (a)(i).

(iii) (*4 marks*)
By using the mean-value theorem, determine how many fixed point iterations would guarantee that $|x_n - x^*|$ is no greater than $10^{-4}$.

(iv) (*3 marks*)
Propose an alternative fixed-point iteration that converges quadratically to a solution, $x^*$, of (1).

(b) Consider the system of equations

$$\left.\begin{aligned} f(x, y) = 4y^2 + 2xy - 5 = 0, \\ g(x, y) = 2y + 3 = 0. \end{aligned}\right\} \tag{2}$$

(i) (*2 marks*)
Determine the exact solution $(x^*, y^*)$ to (2).

(ii) (*8 marks*)
When using Newton's method to solve (2), specify the choice of initial points $(x_0, y_0)$ for which the exact solution of (2) is arrived at after just one iteration.

*Continued...*

3. Let $V_n(x)$, $n = 0, 1, 2, \ldots$, be a sequence of polynomials of degree $n$ that are orthogonal on the interval $(-1, 1)$ with respect to the weight function $w(x) = \sqrt{(1+x)/(1-x)}$. Hence, they satisfy the orthogonality relation

$$\int_{-1}^{1} V_n(x)\, V_m(x) \sqrt{\frac{1+x}{1-x}}\, dx = 0 \quad \text{if} \quad n \neq m.$$

In addition they satisfy the standardisation condition $V_n(1) = 1$ for all $n$. You are *given* the values of the following integrals that involve the weight function $w(x)$:

$$\int_{-1}^{1} w(x)\, \mathrm{d}x = \pi, \qquad \int_{-1}^{1} x w(x)\, \mathrm{d}x = \int_{-1}^{1} x^2 w(x)\, \mathrm{d}x = \frac{\pi}{2}, \qquad \int_{-1}^{1} x^3 w(x)\, \mathrm{d}x = \frac{3\pi}{8}.$$

(a) (*8 marks*)
Determine the polynomials $V_0(x)$, $V_1(x)$ and hence show that $V_2(x) = 4x^2 - 2x - 1$.

(b) (*5 marks*)
Specify the points $x_i$ and weights $w_i$ in the 2-point Gaussian quadrature formula

$$\int_{-1}^{1} f(x) \sqrt{\frac{1+x}{1-x}}\, \mathrm{d}x \approx \sum_{i=1}^{2} w_i\, f(x_i).$$

(c) (*4 marks*)
Use your results of part (b) to find an approximation for the integral

$$\int_{0}^{1} t\, \mathrm{e}^{t}\, \mathrm{d}t.$$

(You are not asked to evaluate this approximation numerically.)

(d) (*8 marks*)
The polynomials $V_n(x)$ have the explicit form

$$V_n(x) = \frac{\cos\left[\left(n + \frac{1}{2}\right) \arccos(x)\right]}{\cos\left[\frac{1}{2} \arccos(x)\right]}. \tag{3}$$

Show that the functions in equation (3) satisfy the recursion relation

$$V_{n+1}(x) + V_{n-1}(x) = f(x)\, V_n(x)$$

where $f(x)$ is a function that you should determine. Infer that the functions in (3) are indeed polynomials of degree $n$.

[*Hint: The trigonometric relation* $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$ *might be helpful.* ]

4. This question (25 marks) is omitted due to requiring non-examinable material.

*End of examination.*