Numerical Analysis 23 – Sol^{n}s

1.(a) (i) **Step 1**

$$A' = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{3}-\frac{1}{4} & \frac{1}{4}-\frac{1}{6} \\ 0 & \frac{1}{4}-\frac{1}{6} & \frac{1}{5}-\frac{1}{9} \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/12 & 1/12 \\ 0 & 1/12 & 4/45 \end{bmatrix}$$

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unseen example, standard method

Step 2

$$A'' = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/12 & 1/12 \\ 0 & 0 & \frac{4}{45}-\frac{1}{12} \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 1/3 & 1 & 1 \end{bmatrix}$$

$$= \frac{3}{12.45} = \frac{1}{180}$$

3

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ 1/3 & 1 & 1 \end{bmatrix} \quad \& \quad U = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1/12 & 1/12 \\ 0 & 0 & 1/180 \end{bmatrix}$$

(ii) we can write

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/12 & 0 \\ 0 & 0 & 1/180 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= D L^T$$

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Similar on problem sheet

$$\det(A) = \det(L)\,\det(D)\,\det(L^T)$$

$$= 1 \cdot \frac{1}{12.180} \cdot 1 = \underline{\underline{\frac{1}{1960}}}$$ (1)

[Q designed to catch errors from part (i)]

(iii) first solve $L\underline{z} = \underline{b}$ by forwards subst.

$$\Rightarrow \left.\begin{array}{l} z_1 = 1 \\ z_2 = 1/2 \\ z_3 = 1 - 1/2 - 1/3 = 1/6 \end{array}\right\}$$ (1)

following lectures

Then $D\underline{y} = \underline{z} \Rightarrow \left.\begin{array}{l} y_1 = 1 \\ y_2 = 6 \\ y_3 = 30 \end{array}\right\}$ (1)

Then $L^T\underline{x} = \underline{y} \Rightarrow$
(by back subst)

$$\begin{array}{l} x_3 = 30 \\ x_2 = -24 \\ x_1 = 1 + 12 + 10 = 3 \end{array}$$ (1)

So $\underline{x} = \begin{pmatrix} 3 \\ -24 \\ 30 \end{pmatrix}$

unseen e.g. but similar examples in problems/lectures

(b)

(i) Step 1: augmented matrix

$$\tilde{A}' = \left[\begin{array}{ccc|c} 1.00 & 0.5 & 0.333 & 1 \\ 0 & 0.0830 & 0.0835 & 0.5 \\ 0 & 0.0835 & 0.0891 & 0.667 \end{array}\right]$$ (1)

Step 2:

$$\tilde{A}'' = \left[\begin{array}{ccc|c} 1 & 0.5 & 0.333 & 1 \\ 0 & 0.0830 & 0.0835 & 0.5 \\ 0 & 0 & 0.00506 & 0.167 \end{array}\right]$$ (1)

Solve to get
$$\left.\begin{aligned} x_3 &= 33.0 \\ x_2 &= -27.1 \\ x_1 &= 3.56 \end{aligned}\right\}$$ (2)

(ii) ~10% error. Determinant is very small so A is nearly singular. (1) This manifests itself in the a''_{33} element which has only 1 sig. fig. (1)

- partial pivoting doesn't make a difference
- Scaled partial pivoting fails to correct. (1)
- Need higher precision arithmetic. (1)

unseen, but used to seeing pivoting as a sol'n to problems

(c) (May be other methods). I consider

$$\begin{bmatrix} 1 & & & & 0 \\ -1 & 1 & & & \\ 0 & -1 & \ddots & & \\ \vdots & 0 & \ddots & \ddots & \\ \vdots & \vdots & & -1 & 1 \end{bmatrix} \underline{x} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

$$\Rightarrow \quad x_1 = b_1$$

$$x_2 = b_2 + x_1 = b_1 + b_2$$

$$x_3 = b_3 + x_2 = b_1 + b_2 + b_3$$

$$\vdots$$

$$x_n = \sum_{i=1}^{n} b_i \quad \text{is general sol}^{\underline{n}}$$

unseen

$$\therefore \quad \underline{x} = \underbrace{\begin{bmatrix} 1 & & & & 0 \\ 1 & 1 & & & \\ 1 & 1 & 1 & & \\ 1 & 1 & \vdots & 1 & \\ \vdots & \vdots & \vdots & \ddots & 1 \end{bmatrix}}_{\text{This is the inverse matrix.}} \underline{b}$$

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2(a) <u>Theorem:</u>

(i) if $g \in C[a,b]$ & $g \in [a,b)$ for $x \in [a,b]$ & if g' exists in $[a,b]$ & $|g'(x)| \le k < 1$ for $\forall x\ [a,b]$ then $\exists$ a unique fixed point $x^* \in (a,b)$ s.t. $x_{n+1} = g(x_n)$ converges to x^* if $x_0 \in (a,b]$.

bookwork defn

3 (for elements not statement)

<u>Here</u>: $[a,b] = [0,1]$

simple application

& $g(x) = 1 - x^2/4 \in [0,1]$ also 1

& $g'(x) = -\frac{1}{2}x$ s.t. $|g'(x)| \le \frac{1}{2} < 1$ 1

for $x \in (0,1]$ ✓

Standard

(ii) fixed point is <u>not</u> at $x^* = 0$ or $x^* = 1$ 1

as $0 < |g'(x^*)| < \frac{1}{2}$ 1 as <u>linear convergence</u> 1

Similar done in examples

(iii) $|x_n - x^*| = |g(x_{n-1}) - g(x^*)|$

$\downarrow$ M.V.T. 1

$= |g'(\xi_{n-1})|\,|x_{n-1} - x^*|$

& $\xi_{n-1} \in (x_{n-1}, x^*)$

& $|g'(\xi_n)| < 1/2$ from above results. 1

$$\therefore \quad e_n < \frac{1}{2} e_{n-1} < \dots < \frac{1}{2^n} e_0$$

& $e_0 = |x_0 - x^*| < 1$ so $\underline{e_n < \frac{1}{2^n}}$ ①

we want $\frac{1}{2^n} < 10^{-4}$ or $2^n > 10^4$

or $n > \frac{4 \ln 10}{\ln 2} \simeq 13.3$

& So an upper bound on # of iterations needed is $\underline{n = 14}$ ①

(iv) We know Newton's method has quadratic convergence & (1) is $f(x) = \frac{x^2}{4} + x - 1$

so

$$x_{n+1} = x_n - \frac{x_n^2/4 + x_n - 1}{x_n/2 + 1}$$ ③

will be quadratically convergent.

similar to examples

simple

(b)(i) Exact soln $\underline{y = -3/2}$, ① $\quad 9 - 3x - 5 = 0$

$\Rightarrow \underline{x = 4/3}$ ①

(ii) Newton's method.

unseen e.g. needing a bit of thought

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} 2y_n & 8y_n + 2x_n \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} f(x_n, y_n) \\ g(x_n, y_n) \end{pmatrix}$$

Jacobian @ (x_n, y_n) ②

(provided $y_n \neq 0$)

$$\Rightarrow \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \frac{1}{4y_n} \begin{pmatrix} 2 & -8y_n - 2x_n \\ 0 & 2y_n \end{pmatrix} \begin{pmatrix} 4y_n^2 + 2x_n y_n - 5 \\ 2y_n + 3 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \frac{1}{4y_n} \begin{pmatrix} 8y_n^2 + 4x_n y_n - 10 - 16y_n^2 - 4x_n y_n - 24y_n - 6x_n \\ 4y_n^2 + 6y_n \end{pmatrix}$$

$$\Rightarrow x_{n+1} = \cancel{x_n} - 2y_n - \cancel{x_n} + 5/2y_n + 4y_n + x_n + 6 + 3x_n/2y_n$$

$$y_{n+1} = -3/2$$ (3)

y_0 is arbitrary (1) since $y_1 = -3/2$ is exact

Then we would want $x_1 = 4/3$ so

$$\frac{4}{3} = \cancel{x_0} + \cancel{2y_0}\ 6 + x_0 + 2y_0 + \frac{5}{2y_0} + \frac{3x_0}{2y_0}$$

$$\Rightarrow x_0\left(1 + \frac{3}{2y_0}\right) = -2y_0 - 5/2y_0 - 14/3$$

$$\Rightarrow x_0 = \frac{(-2y_0 - 5/2y_0 - 14/3)}{(1 + 3/2y_0)}$$ (2)

any $y_0 \notin -3/2$ or 0 & an x_0 ↗

Q3 (a) Let
$$\begin{cases} V_0(x) = a \\ V_1(x) = bx + c \\ V_2(x) = dx^2 + ex + f \end{cases}$$

But $V_n(1) = 1$ so $\underline{a=1}$, $\underline{b+c=1}$, $\underline{d+e+f=1}$

Then $0 = \int_{-1}^{1} V_0 V_1 w \, dx = \frac{\pi}{2} b + \pi c$

$\therefore b + 2c = 0 \Rightarrow \underline{c=-1, b=2}$

So $V_1(x) = 2x - 1$

Next $0 = \int_{-1}^{1} V_0 V_2 w \, dx = \frac{\pi}{2}(d+e) + \pi f$

$\Rightarrow d + e + 2f = 0$ and so $\underline{d+e=2}$, $\underline{f=-1}$

Lastly

$$0 = \int_{-1}^{1} V_1 V_2 w \, dx = 2d \cdot \frac{3\pi}{8} + (2e-d)\frac{\pi}{2} + (2f-e)\frac{\pi}{2} - f\pi$$

$\Rightarrow \underline{e=-2}, \underline{d=4}$

$\therefore V_0 = \underline{1}$, $V_1 = \underline{2x-1}$, $V_2 = \underline{4x^2-2x-1}$

(2) (2) (4)

Parts (a), (b), (c) follow standard questions in class / homeworks + past papers on Gaussian quadrature but $w(x)$ is different

(b) Solve $V_2(x) = 4x^2 - 2x - 1 = 0$ (1)

$$\Rightarrow x = \frac{2 \pm \sqrt{4+16}}{8} = \frac{1}{4} \pm \frac{\sqrt{5}}{4}$$

$$\therefore \underline{x_1 = \frac{1-\sqrt{5}}{4}}, \quad \underline{x_2 = \frac{1+\sqrt{5}}{4}}$$ (2)

Then

$$w_1 = \int_{-1}^{1} w(x)\left(\frac{x-x_2}{x_1-x_2}\right)dx = \frac{\pi/2 - \pi\left(\frac{1+\sqrt{5}}{4}\right)}{\left(-\sqrt{5}/2\right)}$$

$$= \underline{-\frac{\pi}{2\sqrt{5}} + \frac{\pi}{2}}$$

And

$$w_2 = \int_{-1}^{1} w(x)\left(\frac{x-x_1}{x_2-x_1}\right)dx = \frac{\pi/2 - \pi\left(\frac{1-\sqrt{5}}{4}\right)}{\frac{\sqrt{5}}{2}}$$

$$= \underline{\frac{\pi}{2\sqrt{5}} + \frac{\pi}{2}}$$ (2)

(c) $$I = \int_0^1 t\,e^t\,dt = \frac{1}{2}\int_{-1}^{1} \frac{1}{2}(x+1)\,e^{\frac{1}{2}(x+1)}\,dx$$ (1)

after $x = 2t-1$

$$\therefore I = \frac{1}{4}\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\,\sqrt{1-x^2}\,e^{\frac{1}{2}(x+1)}\,dx$$ (2)

$$\simeq \frac{1}{4}\sqrt{1-x_1^2}\,.\,w_1\,.\,e^{\frac{1}{2}(x_1+1)} + \frac{1}{4}\sqrt{1-x_2^2}\,w_2\,e^{\frac{1}{2}(x_2+1)}$$ (1)

(d) So

unseen

$$V_{n+1}(x) + V_{n-1}(x) = \frac{\cos\left[\left(n+\tfrac{1}{2}+1\right)\cos^{-1}x\right] + \cos\left[\left(n+\tfrac{1}{2}-1\right)\cos^{-1}x\right]}{\cos\left(\tfrac{1}{2}\cos^{-1}x\right)}$$

$$= \frac{2\cos\left[\left(n+\tfrac{1}{2}\right)\cos^{-1}x\right].\cos\left(\cos^{-1}x\right)}{\cos\left(\tfrac{1}{2}\cos^{-1}x\right)}$$

$$= \underline{2x\, V_n(x)} \qquad (*)$$

$\therefore f(x) = \underline{2x}$ 3

Obviously $V_0(x) = 1$ 1 is polynomial of deg 0.

Next $V_1(x) = \dfrac{\cos(3\alpha)}{\cos\alpha}$

where $\alpha = \frac{1}{2}\cos^{-1}x$

$$\Rightarrow V_1(x) = \frac{4\cos^3\alpha - 3\cos\alpha}{\cos\alpha}$$

$$= 4\cos^2\alpha - 3 = 4\left(\tfrac{1}{2} + \tfrac{1}{2}\cos 2\alpha\right) - 3$$

$$= 2\cos 2\alpha - 1$$

$$= \underline{2x - 1}$$ 3

is a polynomial of degree 1.

Using $(*)$ & induction $V_n(x)$ is polynomial of 1 deg n.

Q4 (a)(i)

Standard method, done in class & in examples but new scheme

$$y_{i+1} = \alpha_3 y_{i-2} + h\beta_1 f(t_i, y_i) + h\beta_2 f(t_{i-1}, y_{i-1})$$

Local truncation error is:

$$\tau_{i+1} = y(t_i+h) - \alpha_3 y(t_i - 2h) - h\beta_1 y'(t_i) - h\beta_2 y'(t_i - h)$$

$$= y + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' + \dots$$
$$-\alpha_3\left[y - 2hy' + \frac{4h^2}{2}y'' - \frac{8h^3}{6}y''' + \dots\right]$$
$$-h\beta_1 y'$$
$$-h\beta_2\left[y' - hy'' + \frac{h^2}{2}y''' - \dots\right] \quad (2)$$

To minimise error set $\underline{1 - \alpha_3 = 0}$

$\underline{1 + 2\alpha_3 - \beta_1 - \beta_2 = 0}$ & $\underline{1 - 4\alpha_3 + 2\beta_2 = 0}$

$\therefore \underline{\alpha_3 = 1}$, $\underline{\beta_2 = \frac{3}{2}}$ & $\underline{\beta_1 = 3/2}$ (2)

Also $\tau_{i+1} = h^3/6\, y'''\left[1 + \frac{4}{3}\cdot 6 \cdot - 9/2\right]$

$$= \frac{3h^3}{4}y''' = O(h^3)$$

so order of accuracy is $\underline{2}$ (1)

unseen problem, standard method from lectures + examples... here need roots of $z^3=1$...

(ii) Set $f=0$, then $y_{i+1}-y_{i-2}=0$

Consider $y=z^i$. Then $z^3-1=0$ (2)

& 3 roots are $z=1$, $z=\frac{-1\pm i\sqrt{3}}{2}$.

Obviously $|z|=1$ in all cases. (2)

Root condition states that a multistep formula is stable if all the roots of the characteristic eq^n are s.t. $|z|\le 1$ & any root s.t. $|z|=1$ is simple. This is the case we have here so method is stable (1)

(b) $y'(t)=At+B$, $y(0)=0$

$\Rightarrow y=\frac{1}{2}At^2+Bt$ (1)

unseen, but not too difficult

Euler's method: $y_{i+1}=y_i+hf(t_i,y_i)$

$=y_i+h(At_i+B)$ (1)

Now let $y_i=at_i^2+bt_i$ & $t_i=ih$

b

$$a(t_i+h)^2 + b(t_i+h) = at_i^2 + bt_i + Aht_i + hB$$

$$\Rightarrow (2ah)t_i + ah^2 + bh = (Ah)t_i + hB$$

$$\Rightarrow \underline{a = \frac{A}{2}} \quad \& \quad \underline{b = B} - \frac{Ah}{2}$$ (3)

Now the error in Euler's method is

$$y(t_i) - y_i = \frac{1}{2}At_i^2 + Bt_i - \left(\frac{A}{2}t_i^2 + \left(B - \frac{Ah}{2}\right)t_i\right)$$

$$= \underline{\frac{Ah}{2}t_i} \quad \text{as required}$$ (2)

(c) $$y_{i+1} = y_{i-2} + \frac{3}{2}h(At_i + B) + \frac{3}{2}h(A(t_i - h) + B)$$

unseen, follows part (b).

Again let $y_i = at_i^2 + bt_i$ so that

$$a(t_i+h)^2 + b(t_i+h) = a(t_i-2h)^2 + b(t_i-2h) + \frac{3}{2}h(At_i+B) + \frac{3}{2}h(A(t_i-h)+B)$$

$$\Rightarrow 2aht_i + ah^2 + bh = -4aht_i + 4ah^2 - 2hb + 3hAt_i + 3h(B - \tfrac{1}{2}hA)$$

$$\Rightarrow \underline{a = A/2} \quad \& \quad \underline{b = B}$$ (4)

Error is $y(t_i) - y_i = \frac{1}{2}At_i^2 + Bt_i$
$- \frac{1}{2}At_i^2 - Bt_i = 0$

2

This is explained by the fact that the 3rd derivative of the exact sol^n is zero (& higher) so the local truncation error is zero

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