Numerical Analysis exam 2020/21

Solutions

Question 1: (a) The LU-decomposition is well known. The two parameters are a bit unusual. (b) The order of convergence is also standard. The asymptotic error constant is less familiar.

(a)(i)

$$A = \begin{bmatrix} 1 & 3 & \lambda \\ 2 & 4 & 2 \\ 1 & 4 & 1 \end{bmatrix} \qquad \begin{matrix} R_2 - 2R_1 \to R_2 \\ R_3 - R_1 \to R_3 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 & \lambda \\ 0 & -2 & 2-2\lambda \\ 0 & 1 & 1-\lambda \end{bmatrix} \qquad R_3 + \tfrac{1}{2}R_2 \to R_3$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\frac{1}{2} & 1 \end{bmatrix}}_{L} \cdot \underbrace{\begin{bmatrix} 1 & 3 & \lambda \\ 0 & -2 & 2-2\lambda \\ 0 & 0 & 2-2\lambda \end{bmatrix}}_{U}$$

(ii) We have $A\underline{x} = LU\underline{x} = \underline{b}$

Set $U\underline{x} = \underline{y}$, then $L\underline{y} = \underline{b}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ \mu \end{bmatrix} \Rightarrow y_1 = 1,\ y_2 = 4-2 = 2, \quad y_3 = \mu - 1 + 1 = \mu \qquad \underline{y} = \begin{bmatrix} 1 \\ 2 \\ \mu \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & \lambda \\ 0 & -2 & 2-2\lambda \\ 0 & 0 & 2-2\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ \mu \end{bmatrix} \Rightarrow x_3 = \frac{\mu}{2-2\lambda},\ x_2 = -1 + \frac{\mu}{2}$$

$$x_1 = 1 - 3\left(-1 + \frac{\mu}{2}\right) - \lambda\frac{\mu}{2-2\lambda} = 4 - \frac{3}{2}\mu - \frac{\mu\lambda}{2-2\lambda}$$

$$\underline{x} = \begin{bmatrix} 4 - \frac{3}{2}\mu - \frac{\mu\lambda}{2-2\lambda} \\ -1 + \frac{\mu}{2} \\ \frac{\mu}{2-2\lambda} \end{bmatrix}$$

(iii) $\det A = \det L \cdot \det U = -4(1-\lambda)$

one unique solution if $\lambda \neq 1$

If $\lambda = 1$ then $(2-2\lambda)x_3 = \mu$ has no solution if $\mu \neq 0$
It has infinitely many solutions if $\mu = 0$.

(b) (i) $g(x) = \frac{x}{2} + \frac{7}{2x}$

$g'(x) = \frac{1}{2} - \frac{7}{2x^2}$, $g'(\sqrt{7}) = 0$ convergence is faster than linear

$g''(x) = \frac{7}{x^3}$, $g''(\sqrt{7}) = \frac{1}{\sqrt{7}}$ convergence is quadratic

asymptotic error constant $\lambda = \frac{1}{2!} g''(\sqrt{7}) = \frac{1}{2\sqrt{7}}$

(ii) $g(x) = \frac{6x}{7} + \frac{1}{x}$

$g'(x) = \frac{6}{7} - \frac{1}{x^2}$, $g'(\sqrt{7}) = \frac{5}{7}$ linear convergence since value is $0 < \frac{5}{7} < 1$

asymptotic error constant $\lambda = \frac{1}{1!} g'(\sqrt{7}) = 5/7$

(iii) $g(x) = \frac{3x}{8} + \frac{21}{4x} - \frac{49}{8x^3}$

$g'(x) = \frac{3}{8} - \frac{21}{4x^2} + \frac{3 \cdot 49}{8x^4}$, $g'(\sqrt{7}) = \frac{3}{8} - \frac{3}{4} + \frac{3}{8} = 0$ faster than Linear

$g''(x) = \frac{21}{2x^3} - \frac{3 \cdot 49}{2x^5}$, $g''(\sqrt{7}) = \frac{3}{2\sqrt{7}} - \frac{3}{2\sqrt{7}} = 0$ faster than quadratic

$g'''(x) = -\frac{9 \cdot 7}{2 \cdot x^4} + \frac{15 \cdot 49}{2 \cdot x^6}$, $g'''(\sqrt{7}) = \frac{-9+15}{14} \neq 0$ cubic convergence

$\lambda = \frac{1}{3!} g'''(\sqrt{7}) = \frac{1}{6} \cdot \frac{6}{14} = \frac{1}{14}$

Question 2 : Similar questions in homework and previous exams.
~~These polynomials have not been seen before. The main difficulty is part (e).~~

(a) Let $L_0(x) = a$, $L_1(x) = bx + c$, $L_2(x) = dx^2 + ex + f$

Standardisation: $L_n(0) = 1 + n \Rightarrow a = 1, c = 2, f = 3$

$$0 = \int_0^\infty x e^{-x} L_0(x) L_1(x)\,dx = \int_0^\infty e^{-x} x(bx + c)\,dx = 2b + c$$

$$\Rightarrow \quad 0 = 2b + 2, \quad b = -1 \quad L_1(x) = -x + 2, \; L_0(x) = 1$$

$$0 = \int_0^\infty x e^{-x} L_0(x) L_2(x)\,dx = \int_0^\infty x e^{-x}(dx^2 + ex + f)\,dx = 6d + 2e + f$$

$$0 = \int_0^\infty x e^{-x}(-x + 2)(dx^2 + ex + f)\,dx$$

$$= \int_0^\infty (-dx^3 + 2dx^2 - ex^2 + 2ex - fx + 2f)\, x e^{-x}\,dx$$

$$= -24d + 12d - 6e + 4e - 2f + 2f$$

$$= -12d - 2e$$

We have $0 = 6d + 2e + 3$ and $0 = -6d - e \Rightarrow e = -3, d = \frac{1}{2}$

$$L_0(x) = 1, \; L_1(x) = -x + 2, \; L_2(x) = \frac{1}{2}x^2 - 3x + 3$$

(b) Recurrence relation $L_n(x) = (2 - \frac{x}{n}) L_{n-1}(x) - L_{n-2}(x)$

$$L_2(x) = (2 - \tfrac{x}{2})(-x + 2) - 1 = \frac{x^2}{2} - 3x + 3 \quad \text{as before}$$

$$L_3(x) = (2 - \tfrac{x}{3})(\tfrac{x^2}{2} - 3x + 3) - (-x + 2) = -\frac{x^3}{6} + 2x^2 - 6x + 4$$

(c) $$0 = L_2(x) = \frac{x^2}{2} - 3x + 3 \implies x = 3 \pm \sqrt{9-6} = 3 \pm \sqrt{3}$$

$$x_1 = 3 - \sqrt{3} \qquad x_2 = 3 + \sqrt{3} \qquad x_1 - x_2 = -2\sqrt{3}$$

$$\omega_1 = \int_0^\infty x e^{-x} \frac{x - x_2}{x_1 - x_2} dx = \frac{1}{-2\sqrt{3}} \left[2 - (3 + \sqrt{3}\right] = \frac{1+\sqrt{3}}{2\sqrt{3}}$$

$$w_2 = \int_0^\infty x e^{-x} \frac{x - x_1}{x_2 - x_1} dx = \frac{1}{2\sqrt{3}} \left[2 - (3 - \sqrt{3})\right] = \frac{\sqrt{3}-1}{2\sqrt{3}}$$

d) $$\int_0^\infty dx\, e^{-x^2} = \int_0^\infty x e^{-x} \cdot \underbrace{\frac{1}{x} e^{-x^2+x}}_{f(x)} dx \approx \sum_{j=1}^{2} \omega_j f_j$$

$$\approx \frac{1+\sqrt{3}}{2\sqrt{3}} \frac{1}{3-\sqrt{3}} \exp\left(-(3-\sqrt{3})^2 + 3 - \sqrt{3}\right) + \frac{\sqrt{3}-1}{2\sqrt{3}} \frac{1}{3+\sqrt{3}} \exp\left(-(3+\sqrt{3})^2 + 3 + \sqrt{3}\right)$$

e) Consider

$$\int_0^\infty x e^{-x} (-x+2) \sum_{i=0}^{n} \binom{n+1}{n-i} \frac{(-x)^i}{i!} dx \qquad \left(= \int_0^\infty x e^{-x} L_1(x) L_n(x) dx\right)$$

$$= \sum_{i=0}^{n} \frac{(n+1)!}{(n-i)!(i+1)!} \frac{(-1)^i}{i!} \underbrace{\left(-(i+2)! + 2(i+1)!\right)}_{(i+1)!\,(-i-2+2)}$$

term $i=0$ vanishes (hence also $n=0$)

$$= \sum_{i=1}^{n} \frac{(n+1)!}{(n-i)(i+1)!} \frac{(-1)^{i+1}}{i!} i \cdot (i+1)!$$

$$= \sum_{i=1}^{n} \frac{(n+1)!}{(n-i)!(i-1)!} (-1)^{i+1} \qquad j = i-1$$

$$= \sum_{j=0}^{n-1} \frac{(n+1)!}{(n-j-1)!\, j!} (-1)^j = (n+1)n \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j$$

$$= n(n+1)\,(1-1)^{n-1} = 0 \quad \text{unless } n=1$$

Question 3: (a) Euler's method has been practiced in problems class and home work, this question is of medium difficulty.
(b)(i) is standard, (ii) is more difficult due to the roots of the cubic equation (unseen).

(a)(i) $y'' - y' - 6y = 0$, $y(0)=1$, $y'(0)=-2$

Set $u=y'$. Then $\quad y' = u \qquad y(0)=1$
$\quad u' = u + 6y \qquad u(0) = -2$

Apply Euler's method

$$y_{i+1} = y_i + h u_i \qquad y_0 = 1$$
$$u_{i+1} = u_i + h(6y_i + u_i) \qquad u_0 = -2$$

From first line $\quad u_i = \frac{y_{i+1} - y_i}{h} \quad \Longrightarrow$ into second line

$$\frac{y_{i+2} - y_{i+1}}{h} = \frac{y_{i+1} - y_i}{h} + 6h y_i + h\frac{y_{i+1} - y_i}{h} \qquad y_0 = 1 \quad \frac{y_1 - y_0}{h} = -2$$

$$y_{i+2} + y_{i+1}(-2-h) + y_i(1+h-6h^2) = 0 \qquad y_0 = 1 \quad y_1 = 1-2h$$

$\Longrightarrow A = -2-h,\ B = 1+h-6h^2$

(ii) try $y_i = r^i$. Then

$$r^2 + r(-2-h) + (1+h-6h^2) = 0$$

$$(r - (1+3h))(r - (1-2h)) = 0$$

$$r = 1+3h \quad \text{or} \quad r = 1-2h$$

general solution: $\quad y_i = c_1 (1+3h)^i + c_2 (1-2h)^i$

$$1 = y_0 = c_1 + c_2 \qquad 1-2h = y_1 = c_1(1+3h) + c_2(1-2h)$$

$c_1 + c_2 = 1\ ,\ 3c_1 - 2c_2 = -2 \ \Rightarrow\ c_1 = 0\ ,\ c_2 = 1$

$y_i = (1-2h)^i$

Exact solution: try $y(t) = e^{mt} \Rightarrow m^2 - m - 6 = (m-3)(m+2) = 0$

$m = 3$ or $m = -2$

general solution

$y(t) = d_1 e^{3t} + d_2 e^{-2t}$ $\quad 1 = y(0) = d_1 + d_2$ $\quad d_1 = 0\ ,\ d_2 = 1$

$-2 = y'(0) = 3d_1 - 2d_2$

same as before

$\Rightarrow y(t) = e^{-2t}$

error $\quad E = y(t_i) - y_i = e^{-2ih} - (1-2h)^i$

$= 1 - 2ih + \frac{1}{2}(2ih)^2 + \dots - [1 - 2ih + \frac{1}{2}i(i-1)4h^2 + \dots]$

$= 2ih^2 + \dots = O(h^2)$

(b)(i) local truncation error for $\quad y_{i+1} = \alpha_1 y_i + h\beta_0 f_{i+1} + h\beta_2 f_{i-1}$

$T_{i+1} = y(t_i+h) - \alpha_1 y(t_i) - h\beta_0 y'(t_i+h) - h\beta_2 y'(t_i-h)$

$= y + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' + \dots - \alpha_1 y - h\beta_0 [y' + hy'' + \frac{h^2}{2}y''' + \dots]$

$- h\beta_2 [y' - hy'' + \frac{h^2}{2}y''' + \dots]$

$= y(1-\alpha) + y'h(1 - \beta_0 - \beta_2) + \frac{h^2}{2}y''(1 - 2\beta_0 + 2\beta_2)$

$+ \frac{h^3}{6}y'''(1 - 3\beta_0 - 3\beta_1) + \dots$

require

$1 - \alpha = 0 \qquad 1 - \beta_0 - \beta_2 = 0 \qquad 1 - 2\beta_0 + 2\beta_2 = 0$

$\Rightarrow \alpha = 1 \qquad 3 - 4\beta_0 = 0 \qquad \beta_0 = \frac{3}{4}\ \ \beta_1 = \frac{1}{4}$

$T_{i+1} = O(h^3)$ since $1 - 3\beta_0 - 3\beta_1 = -2 \neq 0$ $\Rightarrow$ order of accuracy $p = 2$

(b)(ii) Set $y^i = z^i$ and $h=0$. Then

$$z^3 - \frac{18}{11}z^2 + \frac{9}{11}z - \frac{2}{11} = 0$$

$$11z^3 - 18z^2 + 9z - 2 = 0$$

$z=1$ is a solution : $11 - 18 + 9 - 2 = 0$

polynomial division

$$(11z^3 - 18z^2 + 9z - 2) : (z-1) = (11z^2 - 7z + 2)$$
$$11z^3 - 11z^2$$
$$-7z^2 + 9z$$
$$-7z^2 + 7z$$
$$2z - 2$$

Further roots $z = \dfrac{7 \pm \sqrt{49-88}}{22} = \dfrac{7 \pm i\sqrt{39}}{22}$

with modulus $|z| = \sqrt{\dfrac{49+39}{484}} < 1$

The method is stable:

<u>Root condition:</u> A linear multistep method is stable if all roots of the characteristic polynomial satisfy $|z| \leq 1$ and any root with $|z| = 1$ is simple.

Question 4: (a) practiced in homework (b) bookwork (c) this has been practiced less (d) unseen.

(a) Central difference approximation

$$y'(x) \approx \frac{y(x+h) - y(x-h)}{2h} \qquad y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2}$$

Apply to $\quad y'' - 3y' + 2y = 2x \qquad y(0) = \frac{3}{2} \quad y(1) = \frac{5}{2}$

Divide interval $[0,1]$ into N parts

$h = \frac{1}{N}$, $x_n = nh$, $n = 0, 1, \ldots, N$, $y_n = y(x_n)$

this leads to

$$\frac{1}{h^2}(y_{n+1} - 2y_n + y_{n-1}) - \frac{3}{2h}(y_{n+1} - y_{n-1}) + 2y_n = 2x_n$$

$$y_{n+1}\left(1 - \frac{3}{2}h\right) + y_n(-2 + 2h^2) + y_{n-1}\left(1 + \frac{3}{2}h\right) = h^2 x_n$$

boundary conditions are $y_N = 1$, $y_0 = 1$. This leads to

$$\underbrace{\begin{bmatrix} (-2+2h^2) & (1-\frac{3}{2}h) & & & \\ (1+\frac{3}{2}h) & (-2+2h^2) & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & (-2+2h^2) & (1-\frac{3}{2}h) \\ & & & (1+\frac{3}{2}h) & (-2+2h^2) \end{bmatrix}}_{A} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_{N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} 2x_1 - \frac{3}{2}(1+\frac{3}{2}h) \\ 2x_2 \\ \vdots \\ 2x_{N-2} \\ 2x_{N-1} - \frac{5}{2}(1-\frac{3}{2}h) \end{bmatrix}}_{\underline{b}}$$

(b) $y'' = py' + qy + r \qquad y(a) = \alpha \quad y(b) = \beta$

The linear shooting method solves the boundary value problem by transforming it into the following two initial value problems

$y_1'' = py_1' + qy_1 + r \qquad y_1(a) = \alpha \quad y_1'(a) = 0$

$y_2'' = py_2' + qy_2 \qquad y_2(a) = 0 \quad y_2'(a) = 1$

Then $y_1 + Cy_2$ solves the original problem and C is determined by $\beta = y_1(b) + Cy_2(b)$

Now

(c) $y'' - 3y' + 2y = 2x \qquad y(0) = \frac{3}{2}, \ y(1) = \frac{5}{2}$

this is replaced by

$y_1'' - 3y_1' + 2y_1 = 2x \qquad y_1(0) = \frac{3}{2} \quad y_1'(0) = 0$

$y_2'' - 3y_2 + 2y_2 = 0 \qquad y_2(0) = 0, \ y_2'(0) = 1$

For complementary function try $y(x) = e^{mx}$

$0 = m^2 - 3m + 2 = (m-2)(m-1) \implies m = 1, \ m = 2$

For particular integral try $ax + b \implies -3a + 2(ax+b) = 2x$

$a = 1 \quad b = \frac{3}{2}$

Solution y_1: $\quad y_1(x) = c_1 e^x + c_2 e^{2x} + x + \frac{3}{2}$

$\frac{3}{2} = y_1(0) = c_1 + c_2 + \frac{3}{2} \qquad 0 = y_1'(0) = c_1 + 2c_2 + 1 \implies c_1 = 1, \ c_2 = -1$

Solution y_2: $\quad y_2(x) = d_1 e^x + d_2 e^{2x}$

$0 = y_2(0) = d_1 + d_2 \qquad 1 = y_2'(0) = d_1 + 2d_2 \implies d_2 = 1 \quad d_1 = -1$

$y = y_1 + Cy_2 = (1-c)e^x + (-1+c)e^{2x} + x + \frac{3}{2}$

$y(1) = \frac{5}{2} \implies c = 1$ and $y(x) = x + \frac{3}{2}$

(d) $y''' = py'' + qy' + ry + s \qquad y(a) = \alpha, y'(a) = \beta \quad y(b) = \gamma$

This is replaced by

$$y_1''' = py_1'' + qy_1' + ry_1 + s \qquad y(a) = \alpha, \; y'(a) = \beta, \; y''(a) = 0$$

$$y_2''' = py_2'' + qy_2' + ry_2 \qquad y(a) = 0, \; y'(a) = 0, \; y''(a) = 1$$

Then $y = y_1 + Cy_2$ is a solution and C is determined by

$$y_1(b) + Cy_2(b) = \gamma$$