UNIVERSITY OF BRISTOL

School of Mathematics

Numerical Analysis
MATH 30029
(Paper code MATH–30029)

May/June 2023   2 hours 30 minutes

This paper contains FOUR questions. All answers will be used for assessment.

Calculators of an approved type (permissible for A-level examinations) are permitted.

Candidates may bring four sheets of A4 page (written doubled sided) of handwritten notes into the examination. Candidates must insert these into their answer booklet(s) for collection at the end of the examination.

On this examination, the marking scheme is indicative and is intended only as a guide to the relative weighting of the questions.

Do not turn over until instructed.

1. (a) (*5 marks*)

Prove that any non-singular symmetric matrix, A, can be expressed as $A = LDL^T$ where L is lower triangular with ones on the leading diagonal and D is a diagonal matrix.

(b) (*8 marks*)

Perform an LU decomposition of the matrix

$$A = \begin{bmatrix} 5 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

expressing matrix elements as fractions, not decimals.

Hence, or otherwise, show that $\det(A) = 1$.

(c) (*4 marks*)

Now consider an $n \times n$ matrix defined by

$$A' = \begin{bmatrix} 1 & 0 & \cdots & & 1 \\ 0 & 1 & 0 & \cdots & 1 \\ \vdots & 0 & \ddots & & \vdots \\ & \vdots & & 1 & 1 \\ 1 & 1 & \cdots & 1 & n \end{bmatrix}.$$

Show that it is possible to write $A' = L'L'^T$ where L' is a lower triangular matrix with ones on the leading diagonal.

(d) (*8 marks*)

When $n = 5$, find a matrix P such that $A = PA'P$.

Hence, or otherwise, determine the solution, $\mathbf{x}$, to $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = (4, 0, 2, 1, 1)^T$.

What does this question tell you about using LU decomposition to solve linear systems of equations ?

Continued...

2. (a) (*2 marks*)
Define the *order of convergence* and the *asymptotic error constant* for a sequence $\{x_n\}$ converging to x^* as $n \to \infty$.

(b) (*6 marks*)
Determine the fixed points, x^*, of the iteration

$$x_{n+1} = x_n^2 + x_n - 2$$

and show that it will not converge to the fixed points for initial guesses $x_0 \neq x^*$.

Propose an alternative iterative scheme which will converge to the same fixed points, giving justification.

(c) (*5 marks*)
For a sequence x_n tending to x^* with linear convergence show that Aitken's method results in the approximation

$$x^* \approx x_n - \frac{(x_{n+1} - x_n)^2}{x_{n+2} - 2x_{n+1} + x_n}.$$

(d) (*4 marks*)
Apply Aitken's method to the fixed point iteration $x_{n+1} = x_n + f(x_n)$, expressing your approximation to the fixed point x^* in terms of x_n only.

(e) (*6 marks*)
Consider the following iterative scheme devised for finding the root $x = x^*$ of $f(x) = 0$ where $f'(x^*) \neq 0$:

$$x_{n+1} = x_n - \frac{(f(x_n))^2}{f(x_n + f(x_n)) - f(x_n)}. \tag{1}$$

Determine that the order of convergence is faster than linear.

(f) (*2 marks*)
Explain how (1) is related to Newton's method.

Continued...

3. (a) (*6 marks*)
Explain why
$$I_n = \int_{-1}^{1} \frac{x^n}{1+x^2} \mathrm{d}x$$
is zero if n is a positive odd integer. Show that $I_0 = \pi/2$, $I_2 = 2 - \pi/2$ and find the value of I_4.

(b) (*6 marks*)
Construct a sequence of polynomials $\phi_i(x)$ for $i = 0, 1, 2$ of degree at most i which are orthogonal on $[-1, 1]$ with respect to the weight function $w(x) = 1/(1 + x^2)$. They should satisfy the standardisation condition $\phi_i(1) = 1$.

(c) (*5 marks*)
Determine the values of x_i and w_i for which the approximation
$$\int_{-1}^{1} f(x)w(x)\,\mathrm{d}x \approx \sum_{i=1}^{2} w_i f(x_i)$$
is exact when $f(x)$ is a polynomial of degree 3 or less.

(d) (*8 marks*)
Use the formula derived in part (c) to determine an approximation to
$$I = \int_{-1}^{1} \frac{\ln(1+x)}{1+x^2}\,\mathrm{d}x,$$
simplifying your result.

You should find your answer is not particularly close to its exact value of $I = -0.3716\ldots$. Identify a source of inaccuracy in your approximation and devise a new approximation which still uses part (c) but improves accuracy.

Continued...

4. (a) A linear multistep formula for the differential equation $y' = f(t, y)$ has the form

$$y_{i+1} = \alpha_3 y_{i-2} + h\beta_1 f(t_i, y_i) + h\beta_2 f(t_{i-1}, y_{i-1})$$

(i) *(5 marks)*
Show that the coefficients α_3, β_1 and β_2 for which the local truncation error is as small as possible are given by $\alpha_3 = 1$ and $\beta_1 = \beta_2 = 3/2$. What is the resulting order of accuracy p of the formula?

(ii) *(5 marks)*
Is the resulting formula stable? Cite any theorem that you use.

(b) *(7 marks)*
Consider the problem

$$y'(t) = At + B\,, \quad y(0) = 0, \tag{2}$$

where A and B are constants. Write down the exact solution.

Euler's method replaces problem (2) by a difference equation. You are given that the solution of this difference equation has the form $y_i = at_i^2 + bt_i$ where a and b are constants and $t_i = ih$. Determine the constants a and b and hence show that the error of Euler's method is given by

$$y(t_i) - y_i = \frac{1}{2}Aht_i.$$

(c) *(8 marks)*
Now apply the numerical scheme from part (a),

$$y_{i+1} = y_{i-2} + \frac{3}{2}hf(t_i, y_i) + \frac{3}{2}hf(t_{i-1}, y_{i-1})$$

to problem (2) in part (b) again to obtain a difference equation. Assume that the solution of this difference equation again takes the form $y_i = at_i^2 + bt_i$ with new constants a and b. Determine these constants and calculate the error $y(t_i) - y_i$. How do you explain this last result ?

End of examination.