UNIVERSITY OF BRISTOL

School of Mathematics

NUMERICAL ANALYSIS

MATH 30029

(Paper code MATH–30029)

January 2024   2 hours 30 minutes

Calculators of an approved type (permissible for A-Level examinations) are permitted.

Candidates may bring ONE hand-written sheet of A4 notes, written double sided into the examination. Candidates must insert this sheet into their answer booklet(s) for collection at the end of the examination.

This paper contains FOUR questions. All answers will be used for assessment.

Do not turn over until instructed.

1. (a) (*5 marks*)

   If a $n \times n$ matrix A is symmetric, what property does its elements, a_{ij}, satisfy ? Define the elements, $a_{ij}^{(1)}$ of the reduced matrix $(n-1)\times(n-1)$, $A^{(1)}$, resulting from the first step of Gaussian elimination in terms of a_{ij}. Hence prove that if A is symmetric then $A^{(1)}$ is also symmetric.

   (b) (*6 marks*)

   Perform an LU-decomposition of the matrix

$$A = \begin{bmatrix} 2 & -2 & 2 \\ -2 & -1 & -1 \\ 2 & -1 & -1 \end{bmatrix}.$$

   (c) (*5 marks*)

   Use your answer to part (b) to determine the solution, $\mathbf{x}$, to $A\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = (1, 0, 0)^T$.

   (d) (i) (*4 marks*)

   Show that, provided $|\mu| \neq 1$,

$$(I + \mu P)^{-1} = \frac{(I - \mu P)}{1 - \mu^2}$$

   where I is the Identity matrix and P is any matrix satisfying $P^2 = I$.

   (ii) (*5 marks*)

   Hence, or otherwise, determine A^{-1}, the inverse of the matrix defined in part (b).

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2\. (a) (i) (*5 marks*)

Sketch the curve $f(x) = x^3 - x + \epsilon$ over $-2 < x < 2$. Assume hereafter that $0 < \epsilon < \frac{3}{8}$.

Indicate the position of the three roots of $f(x)$ on your graph and identify simple upper and lower bounds on each of the three roots which could be used to initiate the Bisection method.

(ii) (*5 marks*)

Stating any results used, prove that an initial guess $x_0 \in [0, \frac{1}{2}]$ converges to a fixed point of the iterative scheme $x_{n+1} = x_n^3 + \epsilon$. Prove that the scheme is linearly convergent.

(iii) (*5 marks*)

Taking $x_0 = 0$, explicitly compute x_1, x_2 and x_3 for the scheme in part (ii).

State why Aitken's method can be used in this case. Use the iterates x_1, x_2, x_3 in Aitken's method to produce an approximation which agrees with the exact power series representation of the fixed point,

$$x^* = \epsilon + \epsilon^3 + 3\epsilon^5 + 12\epsilon^7 + O(\epsilon^9),$$

to the order shown.

(iv) (*4 marks*)

Show that if the iteration $x_{n+1} = \dfrac{2x_n^3 - \epsilon}{3x_n^2 - 1}$ converges, then the limit is a root of $f(x)$ and the convergence is at least second order.

(b) Consider the non-linear system of equations

$$y - 3x^2 = -1, \qquad 4xy - 8x^3 = 1.$$

(i) (*4 marks*)

Derive the two-dimensional Newton iterative step for the unknown $\mathbf{x} = (x, y)^T$ and calculate the approximation, $\mathbf{x}^{(1)}$, to a root $\mathbf{x}^*$ after one iteration given the initial guess $\mathbf{x}^{(0)} = (0, -1)^{\mathrm{T}}$.

(ii) (*2 marks*)

Show that, by eliminating y from the system of equations, the problem may be reduced to that considered in part (a) for a suitable choice of parameter which should be explicitly stated.

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3. In this question polynomials $\phi_i(x)$, $i = 0, 1, 2, \ldots$ are defined to be of degree at most i and orthogonal on the interval $[0, \infty)$ with respect to the weight function $w(x) = x\mathrm{e}^{-x}$. They also satisfy the standardisation condition $\phi_i(0) = 1$.

   (a) (*10 marks*)
   Define
   $$I_n = \int_0^\infty x^n \mathrm{e}^{-x}\,\mathrm{d}x.$$
   and show that $I_n = n!$. Hence construct the functions $\phi_0(x)$ and $\phi_1(x)$ and show that $\phi_2(x) = x^2/6 - x + 1$.

   (b) (*5 marks*)
   Determine the values of x_i, w_i, $i = 1, 2$ under which the approximation
   $$\int_0^\infty f(x)w(x)\,\mathrm{d}x \approx \sum_{i=1}^{2} w_i f(x_i)$$
   is exact when $f(x)$ is a polynomial of degree 3 or less.

   (c) (*5 marks*)
   Use the 2-point scheme derived in part (b) to approximate the value of
   $$J = \int_0^\infty x\mathrm{e}^{-x}\cos(x)\,\mathrm{d}x$$
   and compare your result to the exact value of the integral, which you should calculate.

   (d) (*5 marks*)
   You are given that functions $\phi_n(x)$ satisfy the three-term recurrence relation
   $$(n+1)\phi_n(x) = (2n - x)\phi_{n-1}(x) - (n-1)\phi_{n-2}(x)$$
   for $n \geq 2$. Use this to show that
   $$(n+1)\int_0^\infty w(x)\phi_n^2(x)\,\mathrm{d}x = n\int_0^\infty w(x)\phi_{n-1}^2(x)\,\mathrm{d}x.$$

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4. (a) Consider the Adams Moulton formula $y_{i+1} = y_i + \frac{h}{2} f(t_{i+1}, y_{i+1}) + \frac{h}{2} f(t_i, y_i)$ for solving the ordinary differental equation $y' = f(t, y)$.

   (i) (*4 marks*)
   Derive the local truncation error for this formula.

   (ii) (*4 marks*)
   Show that it is stable quoting any theorem that you use.

   (iii) (*4 marks*)
   Using your results in (i) and (ii) deduce the order of the global error produced by applying the formula. Again state the appropriate theorem you use. Is the method convergent?

   (iv) (*6 marks*)
   If $f(t, y) = \lambda y$, find the part of the complex plane in which $\bar{h} = \lambda h$ must be for the Adams Moulton formula to be time stable.

(b) (i) (*3 marks*)
   Explain how an initial-value problem for a second-order differential equation of the form

$$y''(x) = g(x, y(x), y'(x)), \qquad y(a) = \alpha, \qquad y'(a) = \beta,$$

   may be solved numerically for $x > a$ using Euler's method.

   (ii) (*4 marks*)
   Now apply Euler's method to the problem

$$y'' = \cos(y') + 2xy, \qquad y(0) = \alpha, \qquad y'(0) = \beta,$$

   and hence derive a second-order non-linear difference equation of the form

$$y_{i+2} = f(y_{i+1}, y_i)$$

   where y_i approximates $y(ih)$ and f is a function that you are to determine.

End of examination.