UNIVERSITY OF BRISTOL

School of Mathematics

NUMERICAL ANALYSIS

MATH 30029R

(Paper code MATH–30029W)

December 2024   2 hours 30 minutes

Calculators of an approved type (permissible for A-Level examinations) are permitted.

Candidates may bring ONE hand-written sheet of A4 notes, written double sided into the examination. Candidates must insert this sheet into their answer booklet(s) for collection at the end of the examination.

This paper contains FOUR questions. All answers will be used for assessment.

Do not turn over until instructed.

1. (a) Consider the simultaneous linear equations

$$\begin{aligned} 0.01x + 1.6y &= 32.1, \\ x + 0.6y &= 22. \end{aligned}$$

You are given that these equations are satisfied by $x = 10$, $y = 20$.

(i) (*5 marks*)
Using 3 digit precision (calculators will be useful), perform Gaussian elimination on the augmented matrix

$$\widetilde{A} = \begin{bmatrix} 0.01 & 1.6 & 32.1 \\ 1 & 0.6 & 22 \end{bmatrix}$$

and thus evaluate numerical approximations to x, y.

(ii) (*6 marks*)
Show that the use of partial pivoting improves the accuracy of approximations to the exact solution obtained by Gaussian elimination using 3 digit precision.

(b) (*8 marks*)
Perform an LU-decomposition of the 4×4 banded matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

and use your answer to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (1, 1, 1, 1)^T$.

(c) (*6 marks*)
Using permutation matrices only (i.e. without performing a new LU-decomposition), determine the solution, $\mathbf{z}$, of $B\mathbf{z} = \mathbf{b}$ where $B = A^{\mathrm{T}}$.

Continued...

2. (a) (*4 marks*)

   By sketching curves $y = x$ and $y = 1/(1+x^2)$ on the same graph, show that the cubic equation $x^3 + x - 1 = 0$ has just one root, x^*, say, lying in the interval $(0, 1)$.

   (b) (*6 marks*)

   Prove that the iteration

   $$x_{n+1} = \frac{1}{1 + x_n^2}, \qquad n \geq 0$$

   converges to a unique fixed point $x^* \in (0, 1)$ for any initial guess $x_0 \in (0, 1)$, stating any theorems you rely upon. What is the order of convergence of the scheme ?

   (c) (*5 marks*)

   Show that the iteration

   $$x_{n+1} = \frac{1 - x_n}{x_n^2}, \qquad n \geq 0$$

   has the same fixed point, x^*, as in part (b) but that, for $x_0 \neq x^*$, x_n will not converge to x^* as $n \to \infty$.

   (d) (*5 marks*)

   Show that the iteration

   $$x_{n+1} = \frac{2x_n^3 + 1}{3x_n^2 + 1}, \qquad n \geq 0$$

   also has the same fixed point, x^*, as in part (b). Argue why it must converge for x_0 sufficiently close to x^* and demonstrate that convergence is faster than linear.

   (e) (*5 marks*)

   Find the values of α, β and γ which minimise the error as $h \to 0$ in the approximation

   $$\alpha f(x_0 + 2h) + \beta f(x_0 + h) + \gamma f(x_0)$$

   to the derivative $f'(x)$ at $x = x_0$. What is the resulting order of the error ?

Continued...

3. This question concerns the evaluation of

$$I = \int_0^1 \frac{1}{1+x^2}\, dx$$

whose value is *given* to be $\pi/4 = 0.7853981\ldots$

(a) (*4 marks*)
Approximate I using the trapezoidal rule, denoting your answer T_1, and by the Simpson rule, denoting that value S_2. Your answers should be left as fractions.

(b) (*4 marks*)
Now approximate I using the composite trapezoidal rules T_2 and T_4 using two and four subintervals respectively. Hereafter, answers should be expressed to 7 decimal places.

(c) (*5 marks*)
Using only your answers to parts (a) and (b) employ Romberg integration to produce *three* further approximations. You should indicate which of the three approximations is designed to be most accurate and confirm, or otherwise, whether this is the case.

[*You may quote the Romberg formulae. However, should you need it, you are given that the error in the trapezoidal rule, in terms of the step size h, is of the form* $E(h) = a_2h^2 + a_4h^4 + a_6h^6 + \ldots$]

(d) Let $\phi_i(x)$, $i = 0, \ldots$ be a polynomial of degree i, orthogonal with respect to the weight function $w(x) = 1$ on $x \in [0, 1]$ and satisfying the standardisation condition $\phi_i(0) = 1$.

(i) (*6 marks*)
Derive $\phi_0(x)$, $\phi_1(x)$ and show that $\phi_2(x) = 6x^2 - 6x + 1$.

(ii) (*6 marks*)
Calculate the values of x_j and w_j which render the approximation

$$\int_0^1 f(x)\, \mathrm{d}x \approx \sum_{j=1}^{2} w_j f(x_j)$$

exact for any cubic polynomial, $f(x)$.

Use the formula to evaluate (to 7 decimal places) a new approximation to I.

Continued...

4. (a) (*2 marks*)
Consider the initial-value problem

$$\frac{dy}{dt} = f(y, t), \qquad t > 0, \quad y(0) = \alpha.$$

Write down Euler's method for the numerical solution, $y_i \approx y(t_i)$, on a mesh defined by $t_i = ih$, $i = 0, 1, 2, \ldots$ where h is the step size.

(b) (*4 marks*)
Define the local truncation error τ_{i+1} for numerical solutions of initial-value problems and calculate τ_{i+1} for Euler's method.

(c) Consider the problem

$$\frac{dy}{dt} = t + 1, \qquad 0 < t < 1, \quad y(0) = 0.$$

(i) (*2 marks*)
Determine the exact solution, $y(t)$.

(ii) (*6 marks*)
An explicit expression for the solution to the Euler method solution, y_i, can be found by writing $y_i = y_i^h + y_i^p$ where y_i^h is the solution of the homogeneous difference equation which you should find first.
You should use the ansatz

$$y_i^p = at_i^2 + bt_i + c$$

for the particular solution to determine the constants a, b and c.
Hence find y_i.

(iii) (*3 marks*)
Using your answers to parts (c)(i) and (c)(ii) determine the global error, $E_N = y(1) - y_N$, at $t = 1$ where $h = 1/N$. Is your answer consistent with your answer to part (b) ?

(d) Now consider the numerical scheme for solving ordinary differential equations given by

$$y_{i+1} = 4y_i - 3y_{i-1} - 2hf(y_{i-1}, t_{i-1}).$$

(i) (*5 marks*)
Calculate the local truncation error.

(ii) (*3 marks*)
Is the scheme stable ? State any theorem used.

End of examination.