UNIVERSITY OF BRISTOL

School of Mathematics

NUMERICAL ANALYSIS
MATH 30029
(Paper code MATH–30030)

May/June 2023   2 hour 30 minutes

Solutions

Do not turn over until instructed.

Question 1 is new. Question 2 and 3 were set on 2022 resit exam. Question 4 was set for 2022 summer exam but not used due to impact of strikes.

1. (a) (*Set homework problem*)

Since A is non-singular we can write it uniquely as LU where L is lower triangular with $\text{diag}\{L\} = (1, 1, \ldots, 1)^T$ and U is upper triangular. We can write

$$U = D\tilde{U}$$

where $D = \text{diag}\{U\}$ and $\tilde{U}_{ij} = u_{ij}/u_{ii}$ such that $\text{diag}\{\tilde{U}\} = (1, 1, \ldots, 1)^T$. Now $A = LD\tilde{U}$ and $A = A^T$ so

$$LD\tilde{U} = \tilde{U}^T D L^T$$

and since $\tilde{U}^T$ is lower trangular with 1s along the diagonal and LU-decomposition is unique then $\tilde{U}^T = L$ and hence result.

(b) (*Follows standard method described in notes, in set homeworks and on past papers. Fractions are a bit fiddly, but determinant auxiliary question and part (a) should help debugging. I could have done a* 4×4 *to make it simpler, but wanted* 5×5 *for part (d); I could have asked students to put it into form in part (a), but thought this was asking too much.*)

Perform LU decomposition in steps

$$\begin{bmatrix} 5 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & 1 & 0 & 0 & 0 \\ \frac{1}{5} & 0 & 1 & 0 & 0 \\ \frac{1}{5} & 0 & 0 & 1 & 0 \\ \frac{1}{5} & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 1 & 1 & 1 \\ 0 & \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \\ 0 & -\frac{1}{5} & \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} \\ 0 & -\frac{1}{5} & -\frac{1}{5} & \frac{4}{5} & -\frac{1}{5} \\ 0 & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} & \frac{4}{5} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & 1 & 0 & 0 & 0 \\ \frac{1}{5} & -\frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{5} & -\frac{1}{4} & 0 & 1 & 0 \\ \frac{1}{5} & -\frac{1}{4} & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 1 & 1 & 1 \\ 0 & \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ 0 & 0 & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & 1 & 0 & 0 & 0 \\ \frac{1}{5} & -\frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{5} & -\frac{1}{4} & -\frac{1}{3} & 1 & 0 \\ \frac{1}{5} & -\frac{1}{4} & -\frac{1}{3} & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 1 & 1 & 1 \\ 0 & \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & 1 & 0 & 0 & 0 \\ \frac{1}{5} & -\frac{1}{4} & 1 & 0 & 0 \\ \frac{1}{5} & -\frac{1}{4} & -\frac{1}{3} & 1 & 0 \\ \frac{1}{5} & -\frac{1}{4} & -\frac{1}{3} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 5 & 1 & 1 & 1 & 1 \\ 0 & \frac{4}{5} & -\frac{1}{5} & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Finally, $\det\{A\} = 5\frac{4}{5}\frac{3}{4}\frac{2}{3}\frac{1}{2} = 1$ since product of diagonal elements of U.

(c) (*Unseen example, but not difficult*)
Application of LU decomposition is straightforward and done in one pass to give

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & n \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 1 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n-(n-1) \end{bmatrix}.$$

So clearly $A' = L'L'^T$.

(d) (*Permutation matrices in lecture notes, but students won't have seen a question presented like this before. In the second part, students can access solution using (b) even if P is not found, though it is deliberately nasty. I don't expect students to answer the very last part, which might be too vague anyway, but I don't want students to get 100%. I could drop last part.*)
When $n = 5$, A and A' are the same size and the 1st and 5th rows and columns have been switched so the permutation matrix associated with this is

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Pre-multiplication interchanges the 1st and 5th rows and post-multiplication interchanges columns.
Since $P = P^{-1}$ we solve $A'\mathbf{x}' = \mathbf{b}'$ with $\mathbf{x}' = P\mathbf{x}$ and $\mathbf{b}' = P\mathbf{b} = (1, 0, 2, 1, 4)^T$ and $A' = L'L'^T$. Then $\mathbf{L}'\mathbf{y}' = \mathbf{b}'$ is

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix} \mathbf{y}' = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 4 \end{bmatrix}$$

gives $\mathbf{y}' = (1, 0, 2, 1, 0)^T$ and then solve $L'^T\mathbf{x}' = \mathbf{y}'$ or

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 0 & \cdots & 1 \\ 0 & 0 & 1 & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \mathbf{x}' = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

to give $\mathbf{x}' = (1, 0, 2, 1, 0)^T$ and finally $\mathbf{x} = P\mathbf{x}' = (0, 0, 2, 1, 1)^T$.

Without care, LU decomposition of sparse matrices can result in full triangular L and U matrices.

Continued...

2. (a) (*Bookwork: definition in notes*)

If, for $x_n \to x^*$ as $n \to \infty$ then

$$\lim_{n\to\infty} \frac{|x_{n+1} - x^*|}{|x_n - x^*|^\alpha} = \lambda$$

then α is the order of convergence and λ is the asymptotic error constant.

(b) (*Simple example similar to examples in lectures, homeworks and exams.*)

Fixed points are when $x_n \to x^*$ so ${x_n^*}^2 - 2 = 0$ and $x_n^* = \pm\sqrt{2}$. Let $g(x) = x^2 - 2 + x$ s.t. $x_{n+1} = g(x_n)$. Then $g'(x) = 2x + 1$ and

$$|g'(\pm\sqrt{2})| = |1 \pm 2\sqrt{2}| > 1$$

By fixed point theorem, since $|g'(x^*)| \not< 1$, does not converge.

Here, $x^* = \pm\sqrt{2}$ are roots of $f(x) = x^2 - 2$ and $f'(\pm\sqrt{2}) \neq 0$ so Newton's method will converge quadratically. I.e. the scheme

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}$$

is a possible second order scheme.

(c) (*Bookwork: derivation in notes*)

Aitken's method for linear convergence:

$$\frac{x_{n+1} - x^*}{x_n - x^*} \approx \lambda \approx \frac{x_{n+2} - x^*}{x_{n+1} - x^*}$$

from part (a). So

$$(x_{n+1} - x^*)^2 \approx (x_{n+2} - x^*)(x_n - x^*)$$

and then

$$x_{n+1}^2 - 2x_{n+1}x^* \approx x_{n+2}x_n - x^*(x_n + x_{n+2})$$

which gives the result after rearranging.

(d) (*Unseen example*)

If $x_{n+1} = x_n + f(x_n)$ then

$$x_{n+2} = x_{n+1} + f(x_{n+1}) = x_n + f(x_n) + f(x_n + f(x_n)).$$

So from part (c) we have

$$x^* \approx x_n - \frac{[f(x_n)]^2}{x_n - 2x_n - 2f(x_n) + x_n + f(x_n) + f(x_n + f(x_n))}$$

which reduces to the equation given

(e) (*Unseen, difficult and no calculation like this in course, although similar use of Taylor for some results.*)

In this question we need to consider

$$g(x) = x - \frac{[f(x)]^2}{f(x + f(x)) - f(x)}$$

and we want to show that $g'(x^*) = 0$ where $f(x^*) = 0$ but $f'(x^*) \neq 0$. So

$$g' = 1 - \frac{2ff'}{f(x+f) - f} + \frac{f^2(1+f')f'(x+f) - f'}{(f(x+f) - f)^2}.$$

We let $x \to x^*$ and Taylor expand to get

$$g' = 1 - \frac{2ff'}{(f + ff' + \ldots) - f} + \frac{f^2[(1+f)(f' + ff'' + \ldots) - f']}{[(f + ff' + \ldots) - f]^2}.$$

So

$$g' = 1 - 2 + \frac{f^2[f'^2 + ff'' + ff'f'']}{f^2 f'^2} = \frac{(1+f')f''f}{f'^2}$$

and $g'(x^*) = 0$ since $f(x^*) = 0$.

(f) (*Unseen, but forward difference approximation for derivative is in course and easily reverse engineered.*)
If $h = f(x_n)$ is small then

$$f'(x_n) \approx \frac{f(x_n + h) - f(x_n)}{h}$$

which, if we use in, Newton is $x_{n+1} = x_n - f(x_n)/f'(x_n)$ gives the result (1).

Continued...

3. (a) (*Basic calculus, but results required for later parts are given in question, just in case*)
If n is odd, I_n is zero since x^n is odd and $1+x^2$ is even. Then

$$I_0 = \int_{-1}^{1} \frac{1}{1+x^2}\,dx = [\tan^{-1}(x)]_{-1}^{1} = \pi/2$$

$$I_2 = \int_{-1}^{1} \frac{x^2}{1+x^2}\,dx = \int_{-1}^{1} 1 - \frac{1}{1+x^2}\,dx = 2 - \pi/2$$

$$I_4 = \int_{-1}^{1} \frac{x^4}{1+x^2}\,dx = \int_{-1}^{1} x^2 - \frac{x^2}{1+x^2}\,dx = \frac{2}{3} - (2-\pi/2) = -\frac{4}{3} + \pi/2$$

(b) (*Unseen example, but following standard methods in notes, homeworks, previous exams*)
Start with $\phi_0(x) = 1$ is a polynomial of degree 0 s.t $\phi_0(1) = 1$. Next, let $\phi_1(x) = Ax+B$ and require

$$0 = \langle \phi_0, \phi_1 \rangle = A\int_{-1}^{1} \frac{x}{1+x^2}\,dx + B\int_{-1}^{1} \frac{1}{1+x^2}\,dx$$

implying $B = 0$. So $\phi_1(x) = x$ (s.t. $\phi_1(1) = 1$).
Next, let $\phi_2(x) = Cx^2 + Dx + E$ and we require

$$0 = \langle \phi_0, \phi_2 \rangle = C(2-\pi/2) + D.0 + E.\pi/2$$

or $C(2-\pi/2) = -\pi E/2$. Also require

$$0 = \langle \phi_1, \phi_2 \rangle = C.0 + D(2-\pi/2) + E.0$$

so $D = 0$. Finally, $\phi_2(1) = 1$ so $C + E = 1$ and then we find

$$C = \frac{\pi/2}{\pi-2}, \qquad E = \frac{\pi/2-2}{\pi-2}.$$

Therefore

$$\phi_2(x) = \frac{\pi/2}{\pi-2}x^2 + \frac{\pi/2-2}{\pi-2}.$$

(c) (*Continue to follow standard methods*)
For quadrature to be exact for polynomial of degree 3 or less we need $n = 2$, so zeros, $x = x_i$ $(i = 1,2)$ of $\phi_2(x) = 0$ are

$$x_i = \pm\sqrt{\frac{2-\pi/2}{\pi/2}} = \pm\sqrt{\frac{4}{\pi} - 1}.$$

Let $x_1 = +\sqrt{\ }$, $x_2 = -\sqrt{\ }$. Weights are given by:

$$w_1 = \int_{-1}^{1} \frac{(x-x_1)}{(x_1-x_2)} w(x)\,dx$$

where $w(x) = 1/(1+x^2)$ in this question. Here $x_1 - x_2 = 2\sqrt{(4/\pi) - 1}$ and using part (a) again gives

$$w_1 = \pi/4$$

An almost identical calculation gives

$$w_2 = \int_{-1}^{1} \frac{(x-x_2)}{(x_2-x_1)} w(x)\,dx = \pi/4.$$

(d) (*First part is standard, but I'm looking for simplification of the logs. Last part is harder, but similar examples in notes and homework sheets.*)

We have

$$I = \int_{-1}^{1} \frac{\ln(1+x)}{1+x^2}\, dx \approx \frac{\pi}{4}\left[\ln\left(1+\sqrt{\frac{4}{\pi}-1}\right) + \ln\left(1-\sqrt{\frac{4}{\pi}-1}\right)\right] = \frac{\pi}{4}\ln\left(2-\frac{4}{\pi}\right).$$

Using a calculator this gives $I = -0.25\ldots$, and this is far from the exact value.

The primary reason for this is that $\ln(1+x)$ is divergent (but integrable) at $x = -1$. So we can adjust the integral and write

$$I = \int_{-1}^{1} \frac{\ln(1+x)}{1+x^2} - \frac{\ln(1+x)}{2}\, dx + \frac{1}{2}\int_{-1}^{1} \ln(1+x)\, dx.$$

The last integral is

$$C = \frac{1}{2}\left[x\ln x - x\right]_0^2 = \ln(2) - 1.$$

Therefore we have

$$I = \int_{-1}^{1} \frac{\frac{1}{2}(1-x^2)\ln(1+x)}{1+x^2} + C$$

and we apply quadrature to the integral that remains, since $f(x) = \frac{1}{2}(1-x^2)\ln(1+x)$ is now bounded at $x = -1$.

This is enough for full marks.

However, students may want to calculate the result of this

$$I \approx \ln(2) - 1 + \frac{\pi}{8}\left(1-\left(\frac{4}{\pi}-1\right)\right)\ln\left(2-\frac{4}{\pi}\right) \approx -0.397.$$

Continued...

4. (a) (*Standard method following examples in class and homeworks, but a new scheme*)
We start with

$$y_{i+1} = \alpha_3 y_{i-2} + h\beta_1 f(t_i, y_i) + h\beta_2 f(t_{i-1}, y_{i-1}).$$

The local truncation error is

$$\begin{aligned} \tau_{i+1} &= y(t_i + h) - \alpha_3 y(t_i - 2h) - h\beta_1 y'(t_i) - h\beta_2 y'(t_i - h) \\ &= y + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' + \dots \\ &\quad -\alpha_3 \left[y - 2hy' + \frac{4h^2}{2}y'' - \frac{8h^3}{6}y''' + \dots\right] \\ &\quad -hy' - h\beta_2 \left[y' - hy'' + \frac{h^2}{2}y''' - \dots\right]. \end{aligned}$$

In order that the error is minimised we set

$$1 - \alpha_3 = 0, \qquad 1 + 2\alpha_3 - \beta_1 - \beta_2 = 0, \qquad 1 - 4\alpha_3 + 2\beta_2 = 0.$$

Therefore

$$\alpha_3 = 1, \qquad \beta_2 = \frac{3}{2}, \qquad \beta_2 = \frac{3}{2}.$$

Also

$$\tau_{i+1} = \frac{h^3}{6}y''' \left[1 + \frac{4}{3}.6 - \frac{9}{2}\right] = \frac{3}{4}h^3 y''' = O(h^3).$$

So the order of accuracy is 2.

(ii) (*unseen problem, follows standard method described in notes and homework sheets... potential difficulty is need complex cube roots of 1*)
Set $f = 0$ and then $y_{i+1} - y_{i-2} = 0$. Consider $y = z^i$. Then $z^3 - 1 = 0$ and three roots are $z = 1$ and $z = (-1 \pm \sqrt{3})/2$. Obvsioulsy $|z| = 1$ in all cases.
The 'root condition' states that a multistep formula is stable if all the roots of the characteristic equation are s.t. $|z| \leq 1$ and any root s.t. $|z| = 1$ is simple. This is the case here and we conclude that the method is stable.

(b) (*Unseen, but not too difficult*)
We have $y'(t) = At + B$, $y(0) = 0$ and so we can integrate to get $y = \frac{1}{2}At^2 + Bt$.
Euler's method:

$$y_{i+1} = y_i + hf(t_i, y_i) = y_i + h(At_i + B).$$

Now let $y_i = at_i^2 + bt_i$ and $t_i = ih$. Then

$$a(t_i + h)^2 + b(t_i + h) = at_i^2 + bt_i + Aht_i + hB.$$

I.e.

$$(2ah)t_i + ah^2 + bh = (Ah)t_i + hB.$$

Matching terms gives

$$a = A/2, \qquad b = B - Ah/2.$$

The error in Euler's method is

$$y(t_i) - y_i = \frac{A}{2}t_i^2 + Bt_i - \left(\frac{A}{2}t_i^2 + (B - Ah/2)t_i\right) = (Ah/2)t_i$$

as required.

(c) (*Unseen, follows part (b)*)

We have now

$$y_{i+1} = y_{i-2} + \frac{3}{2}h(At_i + B) + \frac{3}{2}h(A(t_i - h) + B).$$

Again, let $y_i = at_i^2 + bt_i$ so that

$$a(t_i + h)^2 + b(t_i + h) = a(t_i - 2h)^2 + b(t_i - 2h) + \frac{3}{2}h(At_i + B) + \frac{3}{2}h(A(t_i - h) + B)$$

I.e.

$$2aht_i + ah^2 + bh = -4aht_i + 4ah_2 - 2hb + 3hAt_i + 3h(B - Ah/2)$$

and matching terms gives

$$a = A/2, \qquad b = B.$$

The error is now

$$y(t_i) - y_i = \frac{A}{2}t_i^2 + Bt_i - \frac{A}{2}t_i^2 - Bt_i = 0.$$

This is explained by the fact that the 3rd derivative (and higher) of the exact solution is zero and so the local truncation error is zero.

End of examination.