UNIVERSITY OF BRISTOL

School of Mathematics

NUMERICAL ANALYSIS
MATH 30029R
(Paper code MATH–30029R)

August 2023   2 hour 30 minutes

Solutions

Do not turn over until instructed.

1. (a) The first step of Gaussian elimination is defined by

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{i1}}{a_{11}} a_{1j}$$

for $i, j = 2, \ldots, n$ and so

$$a_{ji}^{(1)} = a_{ji} - \frac{a_{j1}}{a_{11}} a_{1i} = a_{ij} - \frac{a_{1j}}{a_{11}} a_{i1} = a_{ij}^{(1)}.$$

I.e. the reduced matrix is also symmetric.

(b) Perform LU decomposition in steps

$$\begin{bmatrix} 2 & -2 & 2 \\ -2 & -1 & -1 \\ 2 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 2 \\ 0 & -3 & 1 \\ 0 & 1 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} \equiv LU.$$

(c) Solve in two steps: $L\mathbf{y} = \mathbf{b}$ then $U\mathbf{x} = \mathbf{y}$. So first

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Forward substitution results in $\mathbf{y} = (1, 1, -\frac{2}{3})$. Next

$$\begin{bmatrix} 2 & -2 & 2 \\ 0 & -3 & 1 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -\frac{2}{3} \end{bmatrix}.$$

Back substitution results in $\mathbf{x} = (0, -\frac{1}{4}, \frac{1}{4})$.

(d) (i)

$$(I + \mu P)(I - \mu P) = (1 - \mu^2)I$$

using $P^2 = I$. Rearrange to get result.

(ii)

The "Hence" approach:

$$\mu P = A - I = \begin{bmatrix} 1 & -2 & 2 \\ -2 & -2 & -1 \\ 2 & -1 & -2 \end{bmatrix}$$

and

$$\mu^2 P^2 = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

means that $P^2 = I$ provided $\mu^2 = 9$ and so the formula in the first part gives us

$$A^{-1} = \frac{-1}{8} \begin{bmatrix} 0 & 2 & -2 \\ 2 & 3 & 1 \\ -2 & 1 & 3 \end{bmatrix}.$$

The "or otherwise" means that the solution to part (c) is the first column of A^{-1}. To find 2nd and 3rd columns we need to solve $A\mathbf{x}_2 = (0, 1, 0)^T$ and $A\mathbf{x}_3 = (0, 0, 1)^T$.

Continued...

2. (a) (i) With $\epsilon = 0$ the curve is cubic through $x = -1, 0, +1$ and for $0 < \epsilon < \frac{3}{8}$ this is lifted up the y-axis.

Should be obvious from graph that roots exist in intervals $-2 < x < -1$, $0 < x < \frac{1}{2}$ and $\frac{1}{2} < x < 1$. Can be looser or tighter than this !

To back this up: $f(-2) = -6+\epsilon < 0$, $f(-1) = \epsilon > 0$. $f(0) = \epsilon > 0$, $f(\frac{1}{2}) = -\frac{3}{8}+\epsilon < 0$ $f(1) = \epsilon > 0$.

(ii)

The map is defined by $x_{n+1} = g(x_n)$ where $g(x) = x^3 + \epsilon$. For $0 \le x \le \frac{1}{2}$, $0 < \epsilon \le g(x) \le \frac{1}{2}$. Also

$$|g'(x)| = 3x^2 < 1, \qquad \text{for } 0 \le x \le \tfrac{1}{2}.$$

Hence, by Fixed Point Theorem, there exists a unique fixed point $x^* \in (0, \frac{1}{2})$ s.t. $x_0 \in [0, \frac{1}{2}]$ will converge to x^*.

Since $g'(x) \neq 0$ for $x \neq 0$ then $g'(x^*) \neq 0$ and so the scheme has first order convergence.

(iii)

Simple: $x_1 = \epsilon$, $x_2 = \epsilon + \epsilon^3$ then

$$x_3 = \epsilon + \epsilon^3 + 3\epsilon^5 + 3\epsilon^7 + \epsilon^9.$$

Aitken's method applies to linearly convergent sequences, so can be used here. Quoting from formula (or can deduce directly)

$$x^* \approx x_1 - \frac{(x_2 - x_1)^2}{(x_3 - 2x_2 + x_1)}$$

and, using the numbers supplied this gives

$$x^* \approx \epsilon - \frac{\epsilon^6}{-\epsilon^3 + 3\epsilon^5 + 3\epsilon^7 + \epsilon^9}$$

and we can binomial this

$$x^* \approx \epsilon + \epsilon^3(1 - 3\epsilon^2 - 3\epsilon^4 - \epsilon^6)^{-1}$$

to get

$$x^* \approx \epsilon + \epsilon^3(1 + 3\epsilon^2 + 3\epsilon^4 + \epsilon^6 + 9\epsilon^4 + O(\epsilon^6)) = \epsilon + \epsilon^3 + 3\epsilon^5 + 12\epsilon^7 + O(\epsilon^9).$$

(iv)

Iteration is $x_{n+1} = g(x_n)$ where $g(x) = (2x^3 - \epsilon)/(3x^2 - 1)$. So

$$g'(x) = \frac{6x(x^3 - x + \epsilon)}{(3x^2 - 1)^2}$$

and is zero at the roots of $f(x)$. Thus scheme converges to roots (easy to see) and is second order convergent since $g''(x^*) \neq 0$.

(b) (i) (*Following standard methods and set homework examples. Part (ii) should be easy.*)
Let $f_1(x, y) = y - 3x^2 + 1$ and $f_2(x, y) = 4xy - 8x^3 - 1$.
Jacobian is

$$J = \begin{pmatrix} -6x & 1 \\ 4y - 24x^2 & 4x \end{pmatrix}$$

And inverse is

$$J^{-1} = -\frac{1}{4y} \begin{pmatrix} 4x & -1 \\ -4y + 24x^2 & -6x \end{pmatrix}$$

Then Newton step is

$$\mathbf{x}^{(n+1)} = \mathbf{x}^{(n)} + \frac{1}{4y} \begin{pmatrix} 4x & -1 \\ -4y + 24x^2 & -6x \end{pmatrix} \begin{pmatrix} y - 3x^2 + 1 \\ 4xy - 8x^3 - 1 \end{pmatrix}$$

If $\mathbf{x}^{(0)} = (0, -1)$ then $\mathbf{x}^{(1)} = (-1/4, -1)$.

(ii) We can eliminate between the two equations to get $x^3 - x + \frac{1}{4} = 0$, which is part (a) with $\epsilon = -1/4$.

Continued...

3. (a) $I_0 = 1$ and, for $n \geq 1$ integrating by parts gives

$$I_n = \left[-x^n \mathrm{e}^{-x}\right]_0^\infty + nI_{n-1} = nI_{n-1}.$$

Thus $I_n = n! I_0$ and the result is shown.
We have $\phi_0(x) = 1$, then let $\phi_1(x) = A_1 x + B_1$ requires $B_1 = 1$ and

$$0 = \langle \phi_1, \phi_0 \rangle = \int_0^\infty x\mathrm{e}^{-x}(A_1 x + 1)\, dx$$

and using part (i) gives $A_1 = -I_1/I_2 = -1/2$. So $\phi_1(x) = 1 - x/2$. Next let $\phi_2(x) = A_2 x^2 + B_2 x + C_2$. First $C_2 = 1$, then

$$0 = \langle \phi_2, \phi_0 \rangle = \int_0^\infty x\mathrm{e}^{-x}(A_2 x^2 + B_2 x + 1)\, dx = 6A_2 + 2B_2 + C_2$$

with

$$0 = \langle \phi_2, \phi_1 \rangle = \int_0^\infty x\mathrm{e}^{-x}(A_2 x^2 + B_2 x + 1)(1 - x/2)\, dx = (6-12)A_2 + (2-3)B_2 + (1-1)C_2$$

So $6A_2 + B_2 = 0$ and $6A_2 + 2B_2 = -1$ combine to give $B_2 = -1$ and $A_2 = 1/6$. Thus $\phi_2(x) = x^2/6 - x + 1$.

(b) For 2-point quadrature we define x_i as zeros of $\phi_2(x)$. Solving $x^2 - 6x + 6 = 0$ gives

$$x_1 = 3 - \sqrt{3}, \qquad x_2 = 3 + \sqrt{3}.$$

Then

$$w_1 = \int_0^\infty \frac{(x - x_2)}{(x_1 - x_2)} x\mathrm{e}^{-x}\, dx = \frac{-1}{2\sqrt{3}} \left(2 - (3 + \sqrt{3})\right).$$

This simplifies to

$$w_1 = \frac{1}{2}\left(1 + \frac{\sqrt{3}}{3}\right)$$

Also

$$w_2 = \int_0^\infty \frac{(x - x_1)}{(x_2 - x_1)} x\mathrm{e}^{-x}\, dx = \frac{1}{2\sqrt{3}} \left(2 - (3 - \sqrt{3})\right).$$

This simplifies to

$$w_2 = \frac{1}{2}\left(1 - \frac{\sqrt{3}}{3}\right)$$

(c) A 2-point scheme gives

$$J \approx \frac{1}{2}\left(1 + \frac{\sqrt{3}}{3}\right)\cos(3 - \sqrt{3}) + \frac{1}{2}\left(1 - \frac{\sqrt{3}}{3}\right)\cos(3 + \sqrt{3})$$

or

$$J = \cos(3)\cos(\sqrt{3}) + \sin(3)\sin(\sqrt{3})/\sqrt{3} \approx 0.239$$

The exact answer is found by integrating by parts

$$J = \frac{1}{2}\int_0^\infty x(\mathrm{e}^{(i-1)x} + \mathrm{e}^{(-i-1)x})\, dx$$

to get

$$J = \frac{1}{2} \int_0^\infty \frac{e^{(i-1)x}}{i-1} + \frac{e^{(-i-1)x}}{-i-1} \, dx$$

to give

$$J = \frac{1}{2} \left(\frac{1}{(i-1)^2} + \frac{1}{(i+1)^2} \right) = 0.$$

(d) From the relation given we take inner product with ϕ_n to get

$$(n+1)\langle \phi_n, \phi_n \rangle = -\langle x\phi_n, \phi_{n-1} \rangle$$

using orthogonality. Also taking inner product with ϕ_{n-2} gives

$$0 = -\langle x\phi_{n-1}, \phi_{n-2} \rangle - (n-1)\langle \phi_{n-2}, \phi_{n-2} \rangle$$

Shifting $n-1 \to n$ in the above and substituting into the first relation gives

$$(n+1)\langle \phi_n, \phi_n \rangle = n\langle \phi_{n-1}, \phi_{n-1} \rangle$$

as required.

Continued...

4. (a) (i) (*Standard methods*)

The local truncation error is

$$\begin{aligned}
\tau &= y(t_i+h) - y(t_i) - \frac{h}{2}y'(t_i+h) - \frac{h}{2}y'(t_i) \\
&= y(t_i) + hy'(t_0) + \frac{h^2}{2}y''(t_i) + \frac{h^3}{6}y'''(t_i) + \ldots \\
& -y(t_i) - \frac{h}{2}\left[y'(t_i) + hy''(t_i) + \frac{h^2}{2}y'''(t_i) + \ldots\right] - \frac{h}{2}y'(t_i) \\
&= \left(\frac{h^3}{6} - \frac{h^3}{4}\right)y'''(t_i) + \ldots \\
&= -\frac{h^3}{12}y'''(t_i) + \ldots
\end{aligned}$$

So error is $O(h^3)$.

(ii)

In order to investigate stabilty we let $h \to 0$ and obtain

$$0 = y_{i+1} - y_i$$

The characteristic polynomial is obtained by setting $y_i = z^i$ and so $z = 1$. Using the <u>root condition</u> (a linear multistep method is stable only if all roots of its characteristic polynomial satisfy $z \leq 1$ and any root with $|z| = 1$ has multiplicity one) we see the Adams-Moulton formula is stable.

(iii)

Theorem (Dahlquist): if a linear multistep method has a local truncation error $O(h^{p+1})$ and is stable then the global error is $O(h^p)$. From (i) and (ii) it follows that the global error is $O(h^2)$. The method is convergent since the global error vanishes as $h \to 0$.

(iv)

To investigate time stability we set $f(t, y) = \lambda y$ and obtain

$$y_{i+1} = y_i + \frac{h}{2}\lambda y_{i+1} + \frac{h}{2}\lambda y_i$$

Then set $y_i = z^i$ which reduces above to

$$z\left(1 - \frac{\lambda h}{2}\right) = \left(1 + \frac{\lambda h}{2}\right)$$

or

$$z = \frac{2 + \lambda h}{2 - \lambda h}$$

The time-stability domain in the complex plane is defined by requiring all the roots of the stability polynomial satisfy $|z| < 1$. Hence we need

$$|(-2) - \lambda h| < |2 - \lambda h|$$

which means the distance to (-2) is smaller than distance to $+2$ and so we need $\Re\{\lambda h\} < 0$.

(b) (i)

The second order ODE $y'' = g(x, y, y')$, $y(a) = \alpha$, $y'(a) = \beta$ is transformed into a system of ODEs with

$$y' = u, \qquad u' = g(x, y, u)$$

with $y(a) = \alpha$, $u(a) = \beta$.

Then we apply Euler's method:

$$y_{i+1} = y_i + hu_i, \qquad u_{i+1} = u_i + hg(x_i, y_i, u_i)$$

where $x_i = a + ih$. These equations are iterated from the starting values of $y_0 = \alpha$ and $u_0 = \beta$.

(ii)

Now we choose $g(x, y, y') = \cos(y') + 2xy$ and $a = 0$. Using $u_i = (y_{i+1} - y_i)/h$ we obtain

$$\frac{y_{i+2} - y_{i+1}}{h} = \frac{y_{i+1} - y_i}{h} + h\cos\left(\frac{y_{i+1} - y_i}{h}\right) + 2hx_iy_i$$

which is

$$y_{i+2} = 2y_{i+1} - y_i + h^2\cos\left(\frac{y_{i+1} - y_i}{h}\right) + 2h^2x_iy_i = f(y_{i+1}, y_i)$$

and $x_i = ih$. The initial values are $y_0 = \alpha$ and $(y_1 - y_0)/h = \beta$ or $y_1 = \alpha + h\beta$.

End of examination.