Numerical Analysis — Sheet 1

Rounding, Linear Systems

Please hand in answers to questions 4, 5, 8 to blackboard submission point by 12 noon on Monday 6th October.

1. So-called spherical Bessel functions $j_n(x)$ (used in PDEs to solve wave scattering by spherical obstacles) are most easily defined by the following two-term recurrence relation

$$j_{n+1}(x) = (2n+1)j_n(x)/x - j_{n-1}(x), \qquad n = 1, 2, \ldots$$

with

$$j_0(x) = \frac{\sin(x)}{x}, \qquad j_1(x) = \frac{\sin(x) - x\cos(x)}{x^2}.$$

Use your calculator (15-digit accuracy) to find $j_0(0.3), \ldots, j_{10}(0.3)$ and compare your answers with the exact values below

```
0.985067355537799
0.099102888040642
0.005961524868620
0.000255859769695
0.000008536424265
0.000000232958256
0.000000005378443
0.000000000107607
0.000000000001899
0.000000000000030
0.000000000000000
```

What do you suspect has happened ?

2. In the lectures the linear system $U\mathbf{x} = \mathbf{b}$ was solved for a general $n \times n$ upper triangular matrix U by *backward substitution.* Solve the linear system $L\mathbf{x} = \mathbf{b}$ for a general $n \times n$ lower triangular matrix L by *forward substitution* , i.e. specify the equations that iteratively determine the unknowns x_i for $i = 1, \ldots, n$.

3. Solve the linear system:

$$\begin{aligned} -x_1 + x_2 + x_3 + x_4 &= -2 \\ x_1 - x_2 + x_3 + x_4 &= -2 \\ x_1 + x_2 - x_3 + x_4 &= 2 \\ x_1 + x_2 + x_3 - x_4 &= 2 \end{aligned}$$

by Gaussian elimination. Demonstrate that your solution works.

4. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & \lambda \\ -2 & 1 & -2\lambda \\ \lambda & -2 & 1 \end{bmatrix}$$

where λ is some constant.

(a) By using Gaussian elimination solve the linear system

$$A\mathbf{x} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}, \tag{1}$$

assuming that it has a unique solution.

(b) Calculate the determinant of the matrix A and hence find all possible values of λ for which the linear system (1) has no unique solution.

(c) Additionally, specify the value of λ for which it has no solution, and the value of λ for which it has an infinite number of solutions.

5. An $n \times n$-matrix L_1 is defined by

$$(L_1)_{ij} = \delta_{ij} - l_{i1}\delta_{1j}, \qquad 1 \leq i, j \leq n,$$

where $l_{i1} \in \mathbb{R}$ for $i \geq 2$ and $l_{11} = 0$. Furthermore, δ_{ij} is the Kronecker delta symbol. Confirm that the inverse matrix L_1^{-1} is given by

$$(L_1^{-1})_{ij} = \delta_{ij} + l_{i1}\delta_{1j}, \qquad 1 \leq i, j \leq n$$

by multiplying L_1 into L_1^{-1} and showing the answer is the identity I.

6. (a) Prove that the product of two upper triangular matrices is also an upper triangular matrix.

(b) Prove that the inverse of an upper triangular matrix is also an upper triangular matrix.

(c) Would the same two results above be true for lower triangular matrices ?

7. Consider the $n \times n$ matrix

$$Q = \begin{bmatrix} 1 & q & q^2 & \cdots & q^n \\ q & 1 & q & \dots & q^{n-1} \\ q^2 & q & 1 & \dots & q^{n-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ q^n & q^{n-1} & \dots & q & 1 \end{bmatrix}.$$

(a) For what values of q will Q be singular and why ?

(b) Confirm that the inverse matrix is given by

$$Q^{-1} = \frac{1}{1-q^2} \begin{bmatrix} 1 & -q & 0 & \cdots & 0 \\ -q & 1+q^2 & -q & \ddots & \vdots \\ 0 & -q & 1+q^2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -q \\ 0 & \dots & 0 & -q & 1 \end{bmatrix}$$

(c) Hence solve for $\mathbf{x}$ when $Q\mathbf{x} = \mathbf{b}$ and $\mathbf{b} = (1, 1, \dots, 1)^T$.

(d) Comment on your answer to part (c) in the case $q = 1$.

8. Prove that if A is an $n \times n$ symmetric matrix, then the $(n-1) \times (n-1)$ matrix containing the elements $a_{ij}^{(1)}$, $i, j = 2, \dots, n$ that results from application of the first Gaussian elimination step is also symmetric.

9. (a) Solve $A\mathbf{x} = (1, 1, 9)^T$ for $\mathbf{x} = (x_1, x_2, x_3)$ using Gaussian elimination where

$$A = \begin{bmatrix} 6 & 0 & -1-\epsilon \\ 0 & 3 & -1 \\ 25 & 12 & -8 \end{bmatrix}.$$

and hence demonstrate that the difference in solutions taking $\epsilon = 0$ and taking non-zero $\epsilon \ll 1$ is disproportionately large.

(b) Now employ partial pivoting to determine the solution, but deduce that this doesn't solve the problems identified in (a).

10. Which of the following are true ?

(i) $\det(AB) = \det(A)\det(B)$, (ii) $\det(A+B) = \det(A)+\det(B)$, (iii) $\det(A^T) = \det(A)$

(iv) $\det(A^{-1}) = 1/\det(A)$, (v) $\det(\overline{A}) = \overline{\det(A)}$, (vi) $\det(A^n) = n \det(A)$