Numerical Analysis — Sheet 10

Boundary-value problems

There is no set homework from this sheet.

1. (a) Determine the values of α, β and γ which minimise the error in the following central difference approximation to the second derivative of a function $y(x)$ at x_0

$$y''(x_0) \approx \alpha y(x_0 - h) + \beta y(x_0) + \gamma y(x_0 + h), \tag{1}$$

assuming small h.

(b) If $y(x)$ were a polynomial, up to what degree of polynomial is the approximation (1) exact for any x_0, h ?

(c) Consider the boundary-value problem for $y(x)$ defined by $y''(x) = 1$ for $0 < x < 5$ with $y(0) = y(5) = 0$.

(i) Solve for $y(x)$.

(ii) Apply the approximation proposed in (1) to the boundary-value problem to construct a system of equations for unknowns $y_i \equiv y(i)$, $i = 1, 2, 3, 4$.

(iii) Do you expect the values of $y(i)$ in (ii) to coincide with the exact values. Explain your answer.

2. We want to find an approximate solution of the boundary-value problem (BVP)

$$y''(x) = 1, \qquad y(0) = 0, \quad y(1) = 1.$$

(a) Calculate the exact solution to the BVP above.

(b) Show how by using the finite difference method this leads to system of equations for $y_i \approx y(ih)$ expressed as

$$\begin{aligned} -2y_1 + y_2 &= h^2, \\ y_{i-1} - 2y_i + y_{i+1} &= h^2, \qquad 1 < i < n-1, \\ y_{n-2} - 2y_{n-1} &= h^2 - 1, \end{aligned}$$

where $h = 1/n$.

(c) If $n = 4$ find the resulting solution (y_1, y_2, y_3) of part (a). Calculate the exact solution $y(x)$ and show that $y_i = y(ih)$.

(d) Prove that values of y_i exactly match the exact solution $y(ih)$ for any integer value of n. Explain why.

(e) Now consider the related problem

$$y''(x) = 1, \qquad y(0) = 0, \quad y'(1) = 0. \qquad (**)$$

By introducing what is called a shadow point y_{n+1} which is immediately set equal to y_{n-1} (because of the boundary condition at $x = 1$), show that a possible finite difference approximation is

$$\begin{aligned} -2y_1 + y_2 &= h^2, \\ y_{i-1} - 2y_i + y_{i+1} &= h^2, \qquad 1 < i < n, \\ y_{n-1} - y_n &= \frac{1}{2}h^2. \end{aligned}$$

Prove that the exact solution of $(**)$ also satisfies $y_i = y(ih)$.