Numerical Analysis **Sheet 2**

LU Decomposition, bisection method

Please submit your answers to questions 2, 6 and 7(a) by 12 noon on Monday 13th October to the blackboard submission point

1. Consider the system of equations $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & -4 & 1 \\ -3 & 2 & 3 \\ 4 & -1 & 6 \end{bmatrix}, \qquad \mathbf{b} = (0, 0, 1)^T.$$

   (a) Deduce the three augmented matrices that result from Gaussian elimination when we implement: (i) no pivoting; (ii) partial pivoting; (iii) scaled partial pivoting.

   (b) Determine numerical solutions using methods (i), (ii) and (iii) and 2-digit precision. [Calculators out ! This is time-consuming.]

2. Find an LU decomposition for the matrix

$$A = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

   Use your LU decomposition find the inverse of the matrix A.

3. Find a decomposition of the form

$$PA = LU$$

   for the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 4 & 2 & 3 \\ 2 & -1 & 4 \end{bmatrix}.$$

   Here P is a permutation matrix, L is lower triangular and U is upper triangular.

4. If P_1 and P_2 are orthogonal matrices satisfying $P_i^{-1} = P_i^T$ show that $P = P_1 P_2$ is also an orthogonal matrix.

5. Prove that the LU-decomposition of a matrix A, being such that $\text{diag}\{L\} = (1, 1, \dots, 1)^T$, is unique.

   [*Hint: you may find the results of Sheet1 Q5 useful.*]

6. (a) If A is a non-singular symmetric matrix, show that it is possible to express it in the form $A = LDL^T$ where L is a lower triangular matrix such that $\text{diag}\{L\} = (1, 1, \dots, 1)^T$ and D is a diagonal matrix.

(b) A real matrix, A, is said to be *positive definite* if $\mathbf{x}^T A\mathbf{x} > 0$ for all real $\mathbf{x} \in \mathbb{R}^n$ not identically zero. What conditions on D are needed to ensure the symmetric matrix in part (a) is positive definite ?

(c) Hence show that a symmetric positive definite matrix A can be written in the form $A = QQ^T$ for a Q you should define.

7. (a) Show that the function $f(x) = x^3 - x - 1/4$ has 3 roots, exactly one of which lies in the interval $[1, 2]$.

   [*Hints: you will find that sketching the curve of* $x^3 - x$ *is a useful starting point as well as searching for maxima and minima of the function.*]

   (b) Determine the number of interations of the Bisection method which would assure of finding this root to an accuracy of 10^{-4}.

   (c) Give 2 intervals $[a, b]$ which contain exactly one of the other 2 roots.

8. Consider $f(x) = x^2 - 2$ and perform five iterations of the Bisection method in the starting interval $[1, 2]$ to find an approximation for $\sqrt{2}$. How many digits of your final approximation are guaranteed to be correct?

9. By plotting the two curves $y = \tanh x$ and $y = \mu/x$, where $\mu > 0$ show that there is one positive root x^* to the equation

$$x \tanh x = \mu.$$

   In order to use the Bisection method, one ideally wants tight estimates for upper and lower bounds a, b (respectively) on the value of x^*. For this example I've been able to determine $a = \mu$ if $\mu \geq 1$ and $a = \sqrt{\mu}$ if $\mu < 1$ and $b = \mu + 1$. Can you reproduce these values or do better ?

10. An $n \times n$ matrix A with elements $(A)_{ij} = a_{ij}$ is said to be (row) *diagonally dominant* if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ \neq i}}^{n} |a_{ij}|$$

   (a) [HARD] Consider that A is diagonally dominant and use this to show that the reduced $(n-1) \times (n-1)$ matrix $A^{(1)}$ with elements $a_{ij}^{(1)}$ that results from the first step of Gaussian elimination is also diagonally dominant.

   (b) Deduce that the upper triangular matrix U in the LU decomposition of A is diagonally dominant.