Numerical Analysis **Sheet 3**

Fixed-Point Iteration, Newton-Raphson Method

Please hand in your answer to question 1 by 12 noon Monday 20th October to the Blackboard Assessed Homework 1 submission point.

1. ASSESSED HW PROBLEM

   (a) Consider the simultaneous linear equations

   $$\begin{aligned} 0.01x + 1.6y &= 32.1, \\ x + 0.6y &= 22. \end{aligned}$$

   You are given that these equations are satisfied by $x = 10$, $y = 20$.

   (i) (*5 marks*)
   Using 3 digit precision (calculators will be useful), perform Gaussian elimination on the augmented matrix

   $$\widetilde{A} = \begin{bmatrix} 0.01 & 1.6 & 32.1 \\ 1 & 0.6 & 22 \end{bmatrix}$$

   and thus evaluate numerical approximations to x, y.

   (ii) (*6 marks*)
   Show that the use of partial pivoting improves the accuracy of approximations to the exact solution obtained by Gaussian elimination using 3 digit precision.

   (b) (*8 marks*)
   Perform an LU-decomposition of the 4×4 banded matrix

   $$A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

   and use your answer to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (1, 1, 1, 1)^T$.

   (c) (*6 marks*)
   Using permutation matrices only (i.e. without performing a new LU-decomposition), determine the solution, $\mathbf{z}$, of $B\mathbf{z} = \mathbf{b}$ where $B = A^{\mathrm{T}}$.

2. (a) Show that the function $g(x) = 2^{-x}$ has a unique fixed point x^* in the interval $[1/3, 1]$, and that the fixed point iteration $x_{n+1} = g(x_n)$ with initial point $x_0 \in [1/3, 1]$ converges to this fixed point x^*.

   (b) By using the mean value theorem find an upper bound for the number of iterations of the fixed point method with initial point $x_0 \in [1/3, 1]$ which would be required to find the fixed point x^* to an accuracy of 10^{-4}.

3. In this question we consider solutions to the equation

$$x \tanh x = 1.$$

   (a) By plotting curves of $y = x$ and $y = \coth x$, illustrate that there is just one positive root, x^*, lying in $x^* > 1$.

   (b) Consider the fixed point iteration $x_{n+1} = \coth x_n$ and prove that this converges to the root $x^* \in (1, 2)$ for an initial point $x_0 \in (1, 2)$.

   (c) Determine the order of convergence and find upper and lower bounds on the asymptotic error constant given $x^* \in (1, 2)$.

   (d) Start with $x_0 = 1.5$, tabulate the first 5 iterates (use 5 sig. figs.) and values of $|x_n - x_{n-1}|/|x_{n-1} - x_{n-2}|$ for $n \geq 2$. What do these latter values represent ?

   (e) Devise an iterative scheme which converges to the root more rapidly than the scheme in part (b).

4. Consider the following iterating schemes for calculating $21^{1/3}$. Show that each of the iteration schemes has a fixed point which is a solution of the equation $x^3 = 21$. Rank the methods in order based upon the speed of convergence assuming that $x_0 = 1$ in each case. (You may assume that all four iteration schemes converge to one of their fixed points if $x_0 = 1$.)

   (a) $x_{n+1} = 20x_n/21 + 1/x_n^2$

   (b) $x_{n+1} = x_n - (x_n^3 - 21)/(3x_n^2)$

   (c) $x_{n+1} = (21/x_n)^{1/2}$

   (d) $x_{n+1} = x_n - (x_n^4 - 21x_n)/(x_n^2 - 21)$

5. Using a graphical method investigate the behaviour of the Newton Raphson iteration for calculating the unique root of $f(x) = (4x - 7)/(x - 2)$ given the following starting points i) 1.625 ii) 1.875 iii) 1.5 iv) 1.95 and v) 3.

6. The function $f(x) = \tanh(x)$ has a single zero at $x^* = 0$.

   (a) Write down the Newton iteration for this function and tabulate the iterates x_n, $n = 1, 2, 3$ for $x_0 = 0.5$ and the quotient $|x_n|/|x_{n-1}|^3$.

   (b) Determine the order of convergence, α, for this scheme and the asymptotic error constant, λ. Comment on your result in relation to the tabulated results part (a).

(c) Using Taylor's theorem show that values of x_0 satisfying

$$|x_0| < \sqrt{3/2 \cosh 2x_0}$$

will ensure convergence of the scheme.

(d) Show by a direct approach that the scheme will converge provided the initial guess satisfies $|x_0| > \frac{1}{4}|\sinh 2x_0|$. You are given that the value $x_0 = 1.5$ falls outside this range of values. Calculate iterates x_n for $n = 1, 2$ for $x_0 = 1.5$ to confirm this result.

7. Consider the following algorithm

$$x_{n+1} = x_n - f(x_n)/f^{'}(x_n) - \frac{1}{2}f^{''}(x_n)(f(x_n))^2/(f^{'}(x_n))^3$$

for finding a root of the function $f(x)$. Show that in general this scheme will have at least cubic convergence.

8. Derive the useful estimate

$$\frac{1}{2}\left(x + \frac{a}{x}\right)$$

for approximating $\sqrt{a}$ with an initial guess x by applying Newton Raphson to $f(x) = x^2 - a$. Generalise this to an equivalent formula for $\sqrt[n]{a}$ and use it to estimate $\sqrt[3]{9}$ using an initial guess of $x = 2$.

9. Consider the iterative scheme

$$x_{n+1} = \sqrt{2 - \sqrt{4 - x_n^2}}, \qquad n = 0, 1, 2, \ldots$$

with $x_0 = \sqrt{2} \approx 1.414$ to 4 significant figures.

(a) Show that a fixed point is $x^* = 0$ and that the order of convergence is linear.

(b) Show, by hand, that the scheme is mathematically equivalent to

$$x_{n+1} = \frac{x_n}{\sqrt{2 + \sqrt{4 - x_n^2}}}, \qquad n = 0, 1, 2, \ldots$$

(c) Iterate both schemes up to x_4 retaining only 4 significant figures at each step of your calculation. Tabulate x_n and $2^{n+1}x_n$ for $n = 1, 2, 3, 4$ for each scheme (i.e. 4 columns and 4 rows).

(d) In both schemes $2^{n+1}x_n$ should tend to π. Provide a plausible explanation as to why the first one does not, but the second one does.

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