Numerical Analysis — Sheet 4

Aitken's Δ^2-method, Newton-Raphson in higher dimensions, Lagrange interpolating polynomials

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Please hand in your answers to questions 4 and 7 by 12 noon on Monday 27th October to the blackboard submission point.

1. (a) Rewriting the cubic equation $x^3 + 4x^2 - 10 = 0$ as $x = \frac{1}{2}\sqrt{10 - x^3}$ and using an initial guess $x_0 = 1.5$, use fixed point iteration 6 times to estimate a root of the cubic equation.

   (b) By using Aitken's method, generate 5 better estimates and thereby find the root to 4 decimal places.

2. Suppose that A is any non-singular matrix. Show that one iteration of the Newton method will give the correct solution to the problem $A\mathbf{x} = \mathbf{b}$ for *any initial guess.*

3. If

   $$f(x, y) = ax^2 + by + c, \qquad g(x, y) = dx + e,$$

   where a, b, c, d and e are constants, show that the Newton method will find a solution $(b, d \neq 0)$ to $f = g = 0$ in *two* iterations for any initial guess.

4. Suppose that

   $$f(x, y) = x^2 - y^2, \qquad g(x, y) = 1 + xy.$$

   Find two real roots of the system. Show that if the initial guess is $(x_0, y_0) = (\alpha, \alpha)$ then the Newton method can *never* converge to a root.

5. Consider using the multidimensional Newton's method to find the roots of $f(x, y) = g(x, y) = 0$ where $f(x, y) = \mathrm{e}^x + y$ and $g(x, y) = \mathrm{e}^y - x$.

   (a) Using graphical means, deduce there is only one root (x^*, y^*) satisfying $0 < x^* < 1$, $-\mathrm{e} < y^* < -1$.

   (b) Set up the Newton iterative step.

   (c) Using $(x_0, y_0) = (1, -1)$, find (x_1, y_1) explicitly and (x_2, y_2) numerically to 5 digit precision.

6. Determine the Lagrange interpolating polynomial, $P_2(x)$ of degree 2, which coincides with the curve $f(x) = \mathrm{e}^x$ at $x = 0, 1, 2$. Then show that the maximum bound E_{max}, say, on the error $E = |P_2(x) - f(x)|$ in the interval $0 \leq x \leq 2$ is given by

   $$E_{max} = \frac{e^2\sqrt{3}}{27}.$$

Find the actual maximum error (you are given that solutions of $\mathrm{e}^x - x(1-\mathrm{e})^2 + \frac{1}{2}(\mathrm{e}-3)(\mathrm{e}-1) = 0$ are $x = 0.448307\ldots$ and $1.60644\ldots$) in the interval $0 \le x \le 2$.

7. Let $f(x) = 2^x$.

   (a) Use Lagrange interpolation to find a polynomial $P_2(x)$ of degree at most two that agrees with $f(x)$ at the points $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$. Simplify your result.

   (b) Without using the result of part (a), show that the modulus of the error in the interpolation, $|f(x) - P_2(x)|$, is bounded by $\dfrac{4(\ln(2))^3}{9\sqrt{3}} \approx 0.085$ for all $x \in [0, 2]$.

8. The Gauss-Seidel method is used for approximating solutions to the $n \times n$ linear system of equations $A\mathbf{x} = \mathbf{b}$ under the iterative scheme

$$L\mathbf{x}^{(k+1)} = \mathbf{b} - U\mathbf{x}^{(k)}$$

where $A = L + U$ and $(L)_{ij} = a_{ij}$ if $i \ge j$, zero otherwise and $(U)_{ij} = a_{ij}$ if $i < j$, zero otherwise (that is U is the upper triangular part of A with zeros on and below the leading diagonal, L is the lower triangular part of A with zeros above the leading diagonal.)

   (a) If $\mathbf{x}^*$ is the exact solution of $A\mathbf{x} = \mathbf{b}$ show that the error $\mathbf{e}^{(k)} = \mathbf{x}^{(k)} - \mathbf{x}^*$ at the kth step satisfies

$$L\mathbf{e}^{(k+1)} = -U\mathbf{e}^{(k)}.$$

   (b) If a_{ij} is diagonally dominant (meaning $|a_{ii}| > \sum_{\substack{j=1 \\ \neq i}}^{n} |a_{ij}|$) show that

$$\|\mathbf{e}^{(k+1)}\|_\infty < \|\mathbf{e}^{(k)}\|_\infty$$

   where $\|\mathbf{e}^{(k)}\|_\infty = \max_{1 \le i \le n} |\mathbf{e}_i^{(k)}|$.

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