Numerical Analysis **Sheet 5**

Lagrange Interpolation, Differentiation and Richardson Extrapolation

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Hand in answers to question 5 by 12 noon on Monday 3rd October.

1. [HARD] You are given that $P_n(x)$ is the unique interpolating polynomial of degree at most n satisfying $P_n(x_k) = f(x_k)$ for $k = 0, 1, \ldots, n$.

   Also let $p_{i,j}(x)$, $j \geq i$, denote the polynomial of degree at most $j - i$ which passes through the points $(x_k, f(x_k))$ for $k = i, i+1, \ldots, j$.

   (a) Show, using induction or otherwise, that $p_{i,j}(x)$ satisfy

   $$p_{i,i}(x) = f(x_i), \qquad 0 \leq i \leq n,$$

   and

   $$p_{i,j}(x) = \frac{(x - x_j)p_{i,j-1}(x) - (x - x_i)p_{i+1,j}(x)}{x_i - x_j}, \qquad 0 \leq i < j \leq n.$$

   (b) Use the relations in (a) to devise an algorithm for finding $P_n(x)$.

   (c) Estimate the number of floating point operations (and hence the implied computational speed) required to compute $P_n(x)$ using the algorithm in (b) compared to the definition of the Lagrange interpolating polynomial given in the notes.

2. If $y(x) = ax^2 + bx + c$ (a, b, c constants) show that the central difference approximations

   $$y'(x) \approx \frac{y(x+h) - y(x-h)}{2h}$$

   $$y''(x) \approx \frac{y(x+h) - 2y(x) + y(x-h)}{h^2}$$

   are exact, regardless of h. Why is this ?

3. Consider calculating $f'(1)$ for $f(x) = \mathrm{e}^x$ using the forward difference formula

   $$f'(x_0) \approx \frac{f(x_0 + h) - f(x_0)}{h}.$$

   (a) Without looking at your notes, derive the truncation error incurred when using this formula and hence deduce an *upper bound* on its value when calculating $f'(1)$ expressed in terms of h.

   (b) Using your calculator with only 4 digit precision take $h = 10^{-n}$ with $n = 0, 1, \ldots, 5$ to generate six estimates of $f'(1)$. Find the optimal choice of n resulting in the greatest accuracy.

(c) Derive a formula which enables you to predict h_{opt}, the optimal value of h for greatest accuracy taking into account both truncation and round-off errors.

4. You are given the function values

$$f(0.9) = 0.98$$
$$f(1.0) = 1.00$$
$$f(1.2) = 1.16$$

   (a) Make two separate forward and backward difference approximations to the derivative $f'(1)$.

   (b) Find an approximation to $f'(1)$ that is as accurate as possible given the information provided [*HINT: look for an approximation of the form*

$$f'(x_0) \approx \alpha f(x_0 - h) + \beta f(x_0) + \gamma f(x_0 + 2h)\,.]$$

   (c) The function $f(x) = 1 + 20(x-1)^3$ fits the data provided exactly. Explain why, for this particular function, the approximation to $f'(1)$ determined in (b) is further from the true value than one of the values determined in (a).

   (d) What would the error in the estimate of $f'(1)$ be using the scheme in (b) had the data presented in the question described a quadratic function ?

5. You are told that a function $f(x)$ has the following values (with 7 digit precision)

$$\begin{aligned} f(1.0) &= 2.287355, \\ f(1.1) &= 2.677335, \\ f(1.2) &= 3.094479, \\ f(1.3) &= 3.535581, \\ f(1.4) &= 3.996196. \end{aligned}$$

   (a) Using forward difference and central difference formulae numerically approximate the derivative $f'(1.2)$ with $h = 0.2$ and $h = 0.1$.

   (b) Derive a forward 3-point formula to approximate $f'(x_0)$ using $f(x_0)$, $f(x_0 + h)$ and $f(x_0+2h)$ which is accurate to $O(h^2)$. Use this to find a new approximation to $f'(1.2)$.

   (c) Produce a better estimate of $f'(1.2)$ by using Richardson's extrapolation for the central difference formula. (You are given that the actual value of $f'(1.2)$ is 4.297549.)

6. According to the lecture notes the approximation

$$f''(x_0) \approx \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2}.$$

incurs a truncation error given by $E(h) = -\frac{1}{12}h^2 f^{(iv)}(\xi)$ for some $\xi \in (x_0 - h, x_0 + h)$. Assume that you want to numerically estimate the value of the second derivative of $f(x) = \exp(-x^2)$ at $x = 1$ using the approximation above. Assume further that you can only work with eight digit precision. Estimate the optimal value of h which provides the greatest accuracy.

7. Show that the approximation to $f''(x_0)$ in Q6 incurs an exact error which can be written as a power series in even powers of h.

   Use Richardson extrapolation to show that the approximation

$$\frac{-f(x_0+h)+16f(x_0+h/2)-30f(x_0)+16f(x_0-h/2)-f(x_0-h)}{3h^2}$$

   to $f''(x_0)$ is accurate to $O(h^4)$.

8. Given $f(x_0)$, $f'(x_0+h)$ and $f(x_0-\lambda h)$ where $\lambda = O(1)$ (i.e. $\lambda h = O(h)$) and $h \ll 1$, construct a formula for estimating $f''(x_0)$. What is the error incurred by using this formula ? Show that for $\lambda^2 = 3$, the error is $O(h^2)$.

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