Numerical Analysis — Sheet 6

Integration rules and Romberg

Hand in answers to question 1 by 12 noon on Monday 10th November

1. Let $f(x) = 2^x$ for $x \geq 0$.

   (a) Use Lagrange interpolation to find a polynomial $P_2(x)$ of degree at most two that agrees with $f(x)$ at the points $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$. Simplify your result.

   (b) Without using the result of part (a), show that the modulus of the error in the interpolation, $|f(x) - P_2(x)|$, is bounded by $\dfrac{4(\ln(2))^3}{9\sqrt{3}} \approx 0.085$ for all $x \in [0, 2]$.

   (c) Use Simpson's rule to estimate the value of $I = \displaystyle\int_0^2 2^x \, \mathrm{d}x$ and compare with its exact value, which you should also find. Show that the numerical error falls within the bounds predicted by the theoretical error associated with Simpson's rule.

   (d) Use the composite Simpson rule to find an explicit estimate for the value of I using $h = \frac{1}{2}$ (i.e. $n = 4$). Comment on the size of the error compared to the result in part (c).

   (e) Devise a new approximation to the value of I from the preceeding two estimates in (c), (d) which is designed to be more accurate than both.

2. Consider the integral $I = \displaystyle\int_0^1 \frac{4}{1+x^2} \, \mathrm{d}x$.

   (a) Find the exact value of I.

   (b) Use the composite trapezoidal rule to calculate the values of T_1, T_2, T_4.

   (c) Now evaluate the Romberg iterates $T_2^{(1)}$, $T_4^{(1)}$ and $T_4^{(2)}$.

   (d) What is the result of approximating I by S_2 and S_4 using Simpson's rule ?

3. (a) Use Romberg integration to estimate

   $$\int_0^1 \sin(x) \, \mathrm{d}x$$

   as accurately as possible starting with the trapezoidal estimates T_1, T_2 and T_4. Compare your best estimate with the exact result.

   (b) Do the same for the integral

   $$\int_0^1 x \ln(x) \, \mathrm{d}x.$$

   You will find that the approximation is less accurate in the second case. Can you think of a reason why?

4. An approximation to the definite integral

$$\int_{-1}^{1} x(t)\,\mathrm{d}t$$

is sought that is of the form $a\,x(-1/3) + b\,x(0) + c\,x(1/3)$ where a, b and c are constants. How should they be chosen to ensure that the resulting formula is exact for an arbitrary quadratic polynomial? Also show that your formula is exact for any odd function $x(t)$.

5. For each of the following integrals identify problems implementing standard integration rules and devise strategies for overcoming these difficulties:

(a) $\displaystyle\int_0^1 \ln(\sin x)\,\mathrm{d}x$; (b) $\displaystyle\int_1^\infty \frac{1}{1+x^2}\,\mathrm{d}x$; (c) $\displaystyle\int_0^1 \frac{\mathrm{e}^x}{\sqrt{1-x^2}}\,\mathrm{d}x$; (d) $\displaystyle\int_\pi^\infty \frac{\sin x}{x}\,\mathrm{d}x$.