Numerical Analysis — Sheet 8

Methods for solving Initial Value Problems

Please hand in your answers to question 1 by 12 noon on Monday 24th November

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1. (a) Determine the exact solution to $\dfrac{dy}{dt} = y - \mathrm{e}^{-t}$ for $0 < t \le 1$ with $y(0) = 1$.

   (b) Write down Euler's method for this IVP and show the solution of the resulting difference equation can be written

   $$y_i = \frac{(1 - \mathrm{e}^{-h})(1+h)^i + h\mathrm{e}^{-ih}}{1 + h - \mathrm{e}^{-h}}.$$

   (c) Hence determine the error assuming $ih \ll 1$ where i is the step number and show global error at $t = 1$ is $O(h)$.

2. Derive Taylor's 3rd order method. For the ODE $y'(t) = 2ty(t)$, $t > 0$, write down iterative schemes derived from the application of:

   (a) Euler's method;

   (b) Taylor's 2nd order method;

   (c) Taylor's 3rd order method.

3. Consider the initial value problem

   $$y''' = y' + y, \qquad 0 < t \le 1, \quad \text{with } y(0) = 0,\ y'(0) = 1 \text{ and } y''(0) = 0.$$

   Transform this third-order ODE into a system of first-order ODEs. Apply Euler's methods to obtain a system of first-order difference equations. Finally, show how the system of first-order difference equations can be transformed into one third-order difference equation.

4. In this question you are asked to derive two Runge-Kutta methods of order 2 that are commonly used. Consider an iteration scheme of the form

   $$y_{i+1} = y_i + af(t_i, y_i) + bf(t_i + c, y_i + d). \qquad (1)$$

   [*Note, that the midpoint method that was derived in the lectures corresponds to the choice* $a = 0$, $b = h$, $c = h/2$ *and* $d = hf(t_i, y_i)/2$.]

   (a) Determine the constants a, b, c and d by requiring that $b = a$ and that the scheme in equation (1) differs from Taylor's method of order 2 only in terms of order $\mathcal{O}(h^3)$[1]

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[1] The resulting method is known as modified Euler's method or Heun's method.

(b) Now replace the condition $b = a$ by $b = 3a$ and proceed as before. The resulting method has the property that it minimises a bound on the truncation error.

5. When Runge-Kutta 2nd order (RK2) and 4th order (RK4) schemes are used to solve the initial value problem

$$\frac{dy}{dt} = 2ty, \qquad 0 \le t \le 1, \qquad \text{with } y(0) = 1$$

the following results for $y(1)$ are obtained

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RK2 stepsize  y(1)
    h=0.1     2.70905701,
    h=0.05   2.71598984,
RK4
    h=0.1     2.71827018,
    h=0.05   2.71828108.
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Find the errors of these approximations by comparing the results to the exact solution. Confirm that the h-dependence of the errors agrees with the expected order of accuracy of the methods.