Numerical Analysis **Sheet 9**

Multistep methods: order of accuracy, stability, convergence, time-stability

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Please hand in your answer to question 2, 3(c), 4(a,b) by 12 noon on Monday 1st December.

1. Milne's implicit two-step method[1] is a multi-step method of the form

$$y_{i+1} = y_{i-1} + h\beta_0 f(t_{i+1}, y_{i+1}) + h\beta_1 f(t_i, y_i) + h\beta_2 f(t_{i-1}, y_{i-1}) \,.$$

   Find the coefficients β_0, β_1 and β_2 by requiring that the order of accuracy is as high as possible. Show that the resulting order is four.

2. The backward differentiation formula BD2 is a multistep formula of the form

$$y_{i+1} = \alpha_1 y_i + \alpha_2 y_{i-1} + h\beta_0 f(t_{i+1}, y_{i+1}).$$

   Find the coefficients α_1, α_2 and β_0 by requiring that the local truncation error is as small as possible. What is the resulting order of accuracy of the formula ?

3. Recall, that a formula is convergent if it is consistent and stable. Which of the following linear multistep formulas are convergent ? (You are not required to calculate the exact value p of the order of accuracy: it is sufficient to show that $p > 0$. Note also that $z = 1$ is always a root of the characteristic equation.)

   (a) $y_{i+1} = y_i$

   (b) $y_{i+1} = y_{i-3} + \frac{4}{3}h(f_i + f_{i-1} + f_{i-2})$

   (c) $y_{i+1} = y_{i-1} + \frac{1}{3}h(7f_i - 2f_{i-1} + f_{i-2})$

   (d) $y_{i+1} = \frac{18}{19}(y_i - y_{i-1}) + y_{i-3} + \frac{6}{19}h(f_{i+1} + 4f_i + 4f_{i-2} + f_{i-3})$

   (e) $y_{i+1} = -y_i + y_{i-1} + y_{i-2} + 2h(f_i + f_{i-1})$

4. The RK2 method is given by

$$y_{i+1} = y_i + hf(t_i + h/2, y_i + hk/2), \qquad \text{where } k = f(t_i, y_i).$$

   (a) For the system $y' = \lambda y$, $y(0) = c$ where $\lambda \in \mathbb{C}$ with $\Re\{\lambda\} < 0$ determine the following condition on h for time stability

$$|1 + \lambda h + (\lambda h)^2/2| < 1.$$

   If λ is real and negative, find the range of values of $h > 0$ for which the RK2 method is time-stable.

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[1]Milne, W. E. (1926), "Numerical integration of ordinary differential equations", American Mathematical Monthly, Mathematical Association of America, 33 (9): 455–460.

(b) More generally (but harder) show the boundary of the region of time-stability in the complex $\bar{h}$-plane is defined by

$$\bar{h} \equiv \lambda h = -1 \pm (2e^{i\theta} - 1)^{1/2}, \qquad 0 \le \theta < 2\pi$$

(c) How do the results change if RK2 is replaced by

$$y_{i+1} = y_i + hf(t_{i+1} - h/2, y_{i+1} - hk/2), \qquad \text{where } k = f(t_{i+1}, y_{i+1}) \ ?$$

5. In this question you are asked to consider the behaviour of the linear multistep formula

$$y_{i+1} = (1-\eta)y_i + \eta y_{i-1} + \frac{1}{2}(\eta+3)hf_i + \frac{1}{2}(\eta-1)hf_{i-1} \qquad (*)$$

which is a blend of AB2 ($\eta = 0$) and the central difference formula ($\eta = 1$).

(a) Find, analytically, the range of η over which the formula (*) is stable.

(b) Find, analytically, the order of accuracy of the formula (*). Is it a function of η?

(c) When formula (*) with $\eta = 1/2$ is applied to estimate the value of $y(1)$ for the IVP

$$\frac{dy}{dt} = -100(\, y - \cos(t)\,) - \sin(t), \quad y(0) = 1. \qquad (1)$$

the following results are obtained

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h=0.020  y(1)=-3.232914389551366e+17
h=0.010  y(1)=-3.458381472854445e+13
h=0.008  y(1)= 4.108711365306035e+08
h=0.005  y(1)= 5.403022536173886e-01  error=-5.2e-8
h=0.0025 y(1)= 5.403022928106700e-01  error=-1.3e-8
h=0.001  y(1)= 5.403023037784830e-01  error=-2.1e-9
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where $y(1) = \cos(1) = 0.5403023058681397$ is the exact answer. Demonstrate that once the answers are 'good', they converge at the expected rate.

(d) Note that setting $x(t) = y(t) - \cos(t)$ transforms equation (1) into $x'(t) = -100\,x(t)$. Try to understand the result of part (c) by considering the time-stability polynomial for (*) with $\eta = 1/2$ (and $\lambda = -100$). Use this polynomial to determine the maximum value of h for which both roots of the polynomial satisfy $|z| \le 1$? How does this explain the findings in part (c)?

6. An unforced mass-spring-damper system is governed by the ODE

$$y''(t) + 2\gamma y'(t) + \omega^2 y(t) = 0,$$

where $\gamma > 0$ and ω^2 represent damping and spring constants, and supplied with initial conditions $y(0) = 1$, $y'(0) = 0$. Assume throughout that $\gamma > \omega$: such a system is said to be overdamped.

(a) Show that the general solution to the ODE is

$$y(t) = A\mathrm{e}^{-(\gamma+\sqrt{\gamma^2-\omega^2})t} + B\mathrm{e}^{-(\gamma-\sqrt{\gamma^2-\omega^2})t}$$

where A and B can be determined from the initial conditions. Hence determine that $y(t) \to 0$ as $t \to \infty$.

(b) Write the ODE as a coupled first order system, using $v(t) = y'(t)$ as the second variable.

(c) Applying the Euler method, show that the coupled first order system reduces to

$$y_{i+1} = y_i + hv_i, \qquad v_{i+1} = v_i - 2\gamma h v_i - \omega^2 h y_i$$

with $y_0 = 1$, $v_0 = 0$.

(d) Combine the two Euler equations into a single second order difference equation for y_i and determine starting values y_0 and y_1.

(e) Look for solutions $y_i = Az^i$ and hence show that the pair of values of z are given by

$$z = 1 - h(\gamma \pm \sqrt{\gamma^2 - \omega^2}).$$

(f) Hence determine that numerical solutions using the Euler scheme will become unstable when

$$h > \frac{2}{\gamma + \sqrt{\gamma^2 - \omega^2}}.$$

(g) What feature of the Euler method has this restriction on the step size, h, exposed ?