1. I got the following:

```
0.985067355537799
0.099102888040642
0.005961524868621
0.000255859769704
0.000008536424483
0.000000232964777
0.000000005617358
0.000000010454061
0.000000517085671
0.000029291067308
0.001854583843824
```

   Round off error accumulates to the point where it overruns the exact solutions.

2. The system

$$\begin{bmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}$$

   is solved by forward iteration (assuming that all diagonal elements $l_{ii} \neq 0$):

$$x_1 = \frac{b_1}{l_{11}}, \quad x_2 = \frac{b_2 - l_{21}x_1}{l_{22}}, \quad \text{and} \quad x_i = \frac{b_i - \sum_{k=1}^{i-1} l_{ik}x_k}{l_{ii}}, \; i = 3, \ldots, n.$$

3. The augmented matrix for the linear system of equations is:

$$\begin{bmatrix} -1 & 1 & 1 & 1 & -2 \\ 1 & -1 & 1 & 1 & -2 \\ 1 & 1 & -1 & 1 & 2 \\ 1 & 1 & 1 & -1 & 2 \end{bmatrix}$$

   Row operations of step 1: $R_2 \to R_2 + R_1$; $R_3 \to R_3 + R_1$; $R_4 \to R_4 + R_1$

$$\begin{bmatrix} -1 & 1 & 1 & 1 & -2 \\ 0 & 0 & 2 & 2 & -4 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 2 & 2 & 0 & 0 \end{bmatrix}$$

   The pivot element for step 2 is zero. Hence we interchange rows: $R_2 \leftrightarrow R_3$

$$\begin{bmatrix} -1 & 1 & 1 & 1 & -2 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 & -4 \\ 0 & 2 & 2 & 0 & 0 \end{bmatrix}$$

Row operation for step 2: $R_4 \to R_4 - R_2$

$$\begin{bmatrix} -1 & 1 & 1 & 1 & -2 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 & -4 \\ 0 & 0 & 2 & -2 & 0 \end{bmatrix}$$

Row operation for step 3: $R_4 \to R_4 - R_3$

$$\begin{bmatrix} -1 & 1 & 1 & 1 & -2 \\ 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 & -4 \\ 0 & 0 & 0 & -4 & 4 \end{bmatrix}$$

The resulting system is solved by backward iteration

$$\begin{array}{llll} -4x_4 & = 4 & \to & x_4 = -1 \\ 2x_3 + 2x_4 & = -4 & \to & x_3 = -1 \\ 2x_2 + 2x_4 & = 0 & \to & x_2 = 1 \\ -x_1 + x_2 + x_3 + x_4 & = -2 & \to & x_1 = 1 \end{array}$$

Inserting this result for $\mathbf{x}$ into the original equation shows that the solution is correct.

4. (a) We perform Gaussian elimination by applying row operations to the system

$$\begin{bmatrix} 1 & -1 & \lambda \\ -2 & 1 & -2\lambda \\ \lambda & -2 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

In step 1: $R_2 \to R_2 + 2R_1, \quad R_3 \to R_3 - \lambda R_1,$ and we obtain

$$\begin{bmatrix} 1 & -1 & \lambda \\ 0 & -1 & 0 \\ 0 & \lambda - 2 & 1 - \lambda^2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -2 \\ -1 \\ -1 + 2\lambda \end{bmatrix}$$

In step 2: $R_3 \to R_3 - (2 - \lambda)R_2$, and we obtain

$$\begin{bmatrix} 1 & -1 & \lambda \\ 0 & -1 & 0 \\ 0 & 0 & 1 - \lambda^2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -2 \\ -1 \\ 1 + \lambda \end{bmatrix}$$

The system is now in upper triangular form and can be solved by backward substitution

$$\begin{bmatrix} 1 & -1 & \lambda \\ 0 & -1 & 0 \\ 0 & 0 & 1 - \lambda^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 1 + \lambda \end{bmatrix} \quad \Rightarrow \quad \begin{array}{ll} x_3 & = (1 + \lambda)/(1 - \lambda^2) \\ x_2 & = 1 \\ x_1 & = -2 + x_2 - \lambda x_3 = -(1 + \lambda)/(1 - \lambda^2) \end{array}$$

(b) The determinant of A can be calculated from its original form

$$\det(A) = (1 - 4\lambda) + (-2 + 2\lambda^2) + \lambda\,(4 - \lambda) = \lambda^2 - 1\,,$$

or, more simply, from the upper triangular form, because the row operation of adding the multiple of one row to another does not change the determinant of a matrix. The determinant of A vanishes if $\lambda = \pm 1$, and in these cases the system $A\,\mathbf{x} = \mathbf{b}$ does not have a unique solution.

(c) To determine whether it has no solution or an infinite number of solutions one can look at the final set of equations for x_1, x_2 and x_3.

The equation for x_3 has the form $(1-\lambda^2)\,x_3 = 1+\lambda$. If $\lambda = 1$ then the equation for x_3 is $0 \cdot x_3 = 2$ and has no solution. If $\lambda = -1$ then the equation for x_3 is $0 \cdot x_3 = 0$. This is correct for any value of x_3. Hence one has an infinite number of solutions if $\lambda = -1$ (x_3 is arbitrary, $x_2 = 1$, and $x_1 = x_3 - 1$). bigskip

5. We consider

$$\begin{aligned}(L\,L^{-1})_{il} = \sum_{j=1}^{n} (L)_{ij}(L^{-1})_{jl} &= \sum_{j=1}^{n} (\delta_{ij} - l_{i1}\delta_{1j})(\delta_{jl} + l_{j1}\delta_{1l}) \\ &= \delta_{il} - l_{i1}\delta_{1l} + l_{i1}\delta_{1l} - l_{i1}l_{11}\delta_{1l} \\ &= \delta_{il}\end{aligned}$$

where we used $l_{11} = 0$. It shows that $L\,L^{-1}$ is the identity matrix if $L^{-1}_{jl} = \delta_{jl} + l_{j1}\delta_{1l}$.

Remark: The rule for calculating with the Kronecker delta is: $\sum_{j=1}^{n} f_j \delta_{jk} = f_k$.

6. (a) Let U, V be upper triangular matrices. Then their elements are such that $u_{ij} = 0$ if $i > j$ and $v_{ij} = 0$ if $i > j$. The i, jth element of the product of U and V is

$$(UV)_{ij} = \sum_{k=1}^{n} u_{ik} v_{kj}$$

But if $k < i$ then $u_{ik} = 0$ and if $k > j$ then $v_{kj} = 0$ and so this immediately means that

$$(UV)_{ij} = \begin{cases} \displaystyle\sum_{k=i}^{j} u_{ik} v_{kj}, & \text{if } j \geq i \\ 0, & \text{if } i > j \end{cases}$$

and hence UV is upper triangular

(b) There are different ways to do this (and you may be tempted to develop a proof based on the result of part (a)). Here's my way. We start with the system of equations $U\mathbf{x} = \mathbf{b}$ where $\mathbf{b} = (b_1, b_2, \ldots, b_n)^T$ is arbitrary. We first prove that x_j only depends on b_i for $i = j, \ldots, n$.

We know from backwards substitution that

$$x_n = b_n/u_{nn}, \qquad \begin{aligned} x_{n-1} &= (b_{n-1} - u_{n-1,n}x_n)/u_{n-1,n-1} \\ &= b_{n-1}(1/u_{n-1,n-1}) + b_n(-u_{n-1,n}/(u_{nn}u_{n-1,n-1}))\end{aligned}$$

(so true for $j = n$; and $j = n-1$ for illustration). The general backward substitution step is

$$x_j = \left(b_j - \sum_{k=j+1}^{n} u_{jk} x_k \right) \Big/ u_{jj} \equiv c_j b_j + c_{j+1} b_{j+1} + \ldots + c_n b_n,$$

for some c_j, say. This is the inductive step: if it were true for $j+1$ then it is true for j, and since it was true for n then it is true for all $j < n$. And therefore x_j only depends on b_i for $i = j, \ldots, n$. If we write $\mathbf{x} = U^{-1}\mathbf{b}$ then this must imply that $(U^{-1})_{ij} = 0$ if $i > j$. In other words U^{-1} must be upper triangular.

(c) Yes.

7. (a) Clear immediately that $q = 1$ means all the entries are 1 and so the rank of the matrix is 1 (there is only one linearly independent row). This means there is a zero eigenvalue with multiplicity $n - 1$!!

   We also spot that $q = -1$ gives rise to a matrix in which the rows alternate between $1, -1, 1, -1 \ldots$ and $-1, 1, -1, 1, \ldots$. Thus, all rows are linearly dependent and the rank is also 1.

   Note also, in part (b) we see clearly that the inverse does not exist for $q = \pm 1$.

   (b) We consider multiplying Q^{-1} into Q. The top row of $Q^{-1}Q$ is indeed $(1, 0, \ldots 0)$ (direct calculation). Likewise the bottom row is $(0, \ldots, 0, 1)$. The i, ith entry is (direct calculation)

$$(Q^{-1}Q)_{ii} = \frac{-q.q + (1 + q^2) - q.q}{1 - q^2} = 1$$

   as required. All other off diagonal entries $i \neq j$ are of the form

$$(Q^{-1}Q)_{ij} = \frac{-q^p.q + q^{p-1}(1 + q^2) - q^{p-2}.q}{1 - q^2} = 0$$

   for some $p \geq 2$. Hence the product of $Q^{-1}Q = I$, the Identity.

   (c) The solution is

$$\mathbf{x} = \frac{1}{1 - q^2} \begin{bmatrix} 1 & -q & 0 & \cdots & 0 \\ -q & 1 + q^2 & -q & \ddots & \vdots \\ 0 & -q & 1 + q^2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -q \\ 0 & \cdots & 0 & -q & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/(1 + q) \\ (1 - q)/(1 + q) \\ \vdots \\ (1 - q)/(1 + q) \\ 1/(1 + q) \end{bmatrix}$$

   after some routine algebra.

   (d) Choosing $q = 1$ gives $\mathbf{x} = (\frac{1}{2}, 0, \ldots, 0, \frac{1}{2})^T$. The inverse didn't exist for $q = 1$, but we used it anyway. This solution is certainly *a* solution of the system of equations, but it is *not* unique. E.g. $\mathbf{x} = (\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)^T$ is another solution for $q = 1$ with the RHS given and there will be a total $n - 1$ linearly independent solution vectors. It is interesting to note that as $q \to 1$, the numerical solution selects just one.

8. The first step of Gaussian elimination is defined by

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{i1}a_{1j}}{a_{11}}$$

   and so

$$a_{ji}^{(1)} = a_{ji} - \frac{a_{j1}a_{1i}}{a_{11}} = a_{ij} - \frac{a_{1j}a_{i1}}{a_{11}} = a_{ij}^{(1)}$$

   and the reduced matrix is also symmetric.

9. (a) Applying $R_3 \to R_3 - (25/6)R_1$ as the first step to the augmented matrix gives

$$\begin{bmatrix} 6 & 0 & -1 - \epsilon & 1 \\ 0 & 3 & -1 & 1 \\ 0 & 12 & -\frac{23}{6} + \frac{25}{6}\epsilon & \frac{29}{6} \end{bmatrix}.$$

Then $R_3 \to R_3 - 4R_2$ gives

$$\begin{bmatrix} 6 & 0 & -1-\epsilon & 1 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & \frac{1}{6}+\frac{25}{6}\epsilon & \frac{5}{6} \end{bmatrix}.$$

Solving this by back substitution gives

$$x_3 = \frac{5}{1+25\epsilon}, \qquad x_2 = \frac{2+(25/3)\epsilon}{1+25\epsilon}, \qquad x_1 = \frac{1+5\epsilon}{1+25\epsilon}.$$

When $\epsilon = 0$ we have $\mathbf{x} = (1, 2, 5)^T$. To see the effect of small epsilon use binomial expansion of the denominator to get leading order approximations

$$x_3 \approx 5 - 125\epsilon, \qquad x_2 \approx 2 - (50/3)\epsilon, \qquad x_1 \approx 1 - 20\epsilon$$

(that is, ignoring $O(\epsilon^2)$ terms). E.g. if $\epsilon = 0.01$, the solution is $\mathbf{x} \approx (3.75, 1.83, 0.8)^T$ which is dispropotionately far from the $\epsilon = 0$ result.

(b) Pivoting swaps rows 1 and 3 so that we have

$$\begin{bmatrix} 25 & 12 & -8 & 9 \\ 0 & 3 & -1 & 1 \\ 6 & 0 & -1-\epsilon & 1 \end{bmatrix}.$$

Now eliminate with $R_3 \to R_3 - \frac{6}{25}R_1$ to get

$$\begin{bmatrix} 25 & 12 & -8 & 9 \\ 0 & 3 & -1 & 1 \\ 0 & -\frac{72}{25} & \frac{23}{25}-\epsilon & -\frac{29}{25} \end{bmatrix}.$$

We don't swap rows since 3 is bigger then 72/25. So the second elimination step is $R_3 \to R_3 + \frac{72}{75}R_2$

$$\begin{bmatrix} 25 & 12 & -8 & 9 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & -\frac{1}{25}-\epsilon & -\frac{1}{5} \end{bmatrix}$$

and solving for x_3 gives the same as before, etc etc...

So partial pivoting has not obviously eliminated the sensitivity of the solution to changes in one of the elements of the matrix (and scaled pivoting makes no difference either).

This is an example of an *ill-conditioned matrix.*

10. All but (ii) and (vi) are true. The corrected version of (vi) is $\det(A^n) = (\det(A))^n$. See `https://en.wikipedia.org/wiki/Determinant` which has a nice short summary of the history of Determinants.