Boundary value problems

---

1. (a) This is from earlier in the course, but a good reminder:

$$\begin{aligned}\alpha y(x_0-h)+\beta y(x_0)+\gamma y(x_0+h) &= \\ &\alpha\left[y(x_0)-hy'(x_0)+\frac{h^2}{2}y''(x_0)-\frac{h^3}{6}y'''(x_0)+\frac{h^4}{24}y^{(iv)}(\xi_1)\right] \\ &+\beta y(x_0) \\ &+\gamma\left[y(x_0)+hy'(x_0)+\frac{h^2}{2}y''(x_0)+\frac{h^3}{6}y'''(x_0)+\frac{h^4}{24}y^{(iv)}(\xi_2)\right]\end{aligned}$$

where $\xi_1 \in (x_0-h, x_0)$ and $\xi_2 \in (x_0, x_0+h)$. To make this the same as $y''(x_0)$ we want to eliminate coefficient of $y(x_0)$:

$$\alpha+\beta+\gamma=0$$

and $y'(x_0)$:

$$-\alpha h+\gamma h=0$$

but coefficient of $y''(x_0)$ should be one:

$$\alpha h^2/2+\gamma h^2/2=1$$

and solving these three equations gives $\alpha=\gamma=1/h^2$ and $\beta=-2/h^2$. The coefficient of $y'''(x_0)$ is

$$-\alpha h^3/6+\gamma h^3/6$$

which also vanishes. The coefficient of $y^{(iv)}(x_0)$ is

$$\alpha h^4/24+\gamma h^4/24$$

which is not zero. This means the error is $O(h^2)$ (we go a bit further in the notes, not necessary here) and so

$$y''(x_0) \approx \frac{y(x_0-h)-2y(x_0)+y(x_0+h)}{h^2}+O(h^2)$$

(b) Since the error is proportional to $y^{(iv)}(\xi)$ for some $\xi \in (x_0-h, x_0+h)$ if $y^{(iv)}(x) \equiv 0$ then there is no error and approximation is exact. This means it is exact for polynomials of degree 3 or less.

(c) (i) Solving $y''(x)=1$ gives $y(x)=x^2/2+Ax+B$ and if $y(0)=0$ then $B=0$. If $y(5)=0$ then $A=-5/2$. So the solution is

$$y(x)=x(x-5)/2$$

(ii) Use formula with $h = 1$ at $x_0 = 1$:

$$y''(1) = 1 \approx y(1-1) - 2y(1) + y(1+1) = -2y_1 + y_2$$

after using $y(0) = 0$. And at $x_0 = 2$:

$$y''(2) = 1 \approx y(2-1) - 2y(2) + y(2+1) = y_1 - 2y_2 + y_3$$

And at $x_0 = 3$:

$$y''(3) = 1 \approx y(3-1) - 2y(3) + y(3+1) = y_2 - 2y_3 + y_4$$

And at $x_0 = 4$:

$$y''(4) = 1 \approx y(4-1) - 2y(4) + y(4+1) = y_3 - 2y_4$$

after using $y(5) = 0$. So the system of equations for y_i, $i = 1, 2, 3, 4$ can be written

$$\begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

(iii) We would have to solve this using Gaussian Elimination (why not have a go ?). It turns out that the answers are $y_1 = -2$, $y_2 = -3$, $y_3 = -3$, $y_4 = -2$. These are the values of $y(1)$, $y(2)$, $y(3)$, $y(4)$ from the exact solution.
Did we expect this ? Yes, because the exact solution of the ODE is a quadratic function and the central difference approximation for the second derivative we have used is exact for quadratics.

2. (a) Integrating up $y''(x) = 1$ gives a general solution $y(x) = x^2/2 + Ax + B$ and applying $y(0) = 0$ means $B = 0$ and then $y(1) = 1$ means $A = -1/2$. Thus the solution is

$$y(x) = \frac{1}{2}x(x+1)$$

(b) We apply the finite difference method to the BVP. This means the interval $[0, 1]$ is divided into n equal subintervals.
The mesh points are $x_i = ih$, $i = 0, 1, \ldots, n$, and the step size is $h = 1/n$.
The second derivative at the mesh points is approximated by the central difference approximation

$$y''(x_i) = \frac{y(x_i + h) - 2y(x_i) + y(x_i - h)}{h^2} - \frac{h^2}{12} y^{(4)}(\xi_i), \tag{1}$$

where $x_i - h < \xi_i < x_i + h$.
We let y_i denote the approximation to $y(x_i) = y(ih)$. We insert the approximation (1) for the second derivative in the BVP $y''(x) = 1$, neglect the error term, and thus obtain

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = 1, \; 1 \le i \le n-1, \qquad y_0 = 0, \quad y_n = 1.$$

Since we know $y(0) = 0$ and $y(1) = 1$ we insert $y_0 = 0$ and $y_n = 1$ into the first and last of these equations, respectively, so that we end up with

$$\begin{aligned} -2y_1 + y_2 &= h^2, \\ y_{i-1} - 2y_i + y_{i+1} &= h^2, \qquad 1 < i < n-1, \\ y_{n-2} - 2y_{n-1} &= h^2 - 1, \end{aligned}$$

as required.

(c) If $n = 4$ we obtain a set of three linear equations for the unknowns y_1, y_2, and y_3. They can be written in matrix form as ($h = 1/4$)

$$\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{16} \\ \frac{1}{16} \\ -\frac{15}{16} \end{bmatrix}$$

The solutions can be found by Gaussian elimination and are given by

$$y_1 = \frac{5}{32}, \quad y_2 = \frac{12}{32}, \quad y_3 = \frac{21}{32}.$$

We can see that these values agree with the exact solution at the mesh point $x_i = ih$ since $y(ih) = i(i+4)/32$.

(d) We are asked to show that the solution of the finite difference method for general n is given by $y_i = y(ih) = (ih)^2/2 + ih/2$ where $i = 0, 1, \ldots, n$. This can be shown by demonstrating that it satisfies the difference equation of the finite difference method

$$\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} = \frac{h^2[(i+1)^2 - 2i^2 + (i-1)^2] + h[(i+1) - 2i + (i-1)]}{2h^2} = 1.$$

This is correct for $1 \le i \le n-1$. The boundary conditions are also correct $y_0 = 0$ and $y_n = 1$.

The central difference approximation in equation (1) has an error term that is proportional to the fourth derivative of the function y (at some point ξ_i in the interval). The exact solution in our case is a quadratic polynomial, and hence the error term vanishes and the approximation is exact.