Numerical Analysis **Solutions for sheet 3**

Fixed-Point Iteration, Newton-Raphson Method

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1. ASSESSED HW MODEL SOLUTION

   (a) (i) Perform row operation $R_2 \to R_2 - 100R_1$ gives

   $$\begin{bmatrix} 0.01 & 1.6 & 32.1 \\ 0 & -159 & -3.19 \times 10^3 \end{bmatrix}$$

   since using 3 digits, so $22 - 3210 = 3188$ rounds to -3190. Then $y = -3190/159 = 20.1$ after rounding and $0.01x = 32.1 - 32.2 = -0.1$ after rounding meaning that $x = -0.1/0.01 = -10$. This is way out.

   (ii) Partial pivoting requires us to swap rows before eliminating. So

   $$\begin{bmatrix} 1 & 0.6 & 22 \\ 0.01 & 1.6 & 32.1 \end{bmatrix}$$

   and doing $R_2 \to R_2 - 0.01R_1$ gives

   $$\begin{bmatrix} 1 & 0.6 & 22 \\ 0 & 1.59 & 31.9 \end{bmatrix}$$

   after rounding. Now $y = 31.9/1.59 = 20.1$ after rounding and back substituting $x = 22 - 12.1 = 9.9$. This is much better.

(b) Perform LU decomposition in steps

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & \frac{3}{2} & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & \frac{3}{2} & -1 \\ 0 & 0 & 0 & \frac{5}{3} \end{bmatrix} = LU.$$

Solve in two steps: $L\mathbf{y} = \mathbf{b}$ then $U\mathbf{x} = \mathbf{y}$. So first

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & \frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Forward substitution easily gives $\mathbf{y} = (1, 0, 1, \frac{1}{3})^T$. Next

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & \frac{3}{2} & -1 \\ 0 & 0 & 0 & \frac{5}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ \frac{1}{3} \end{bmatrix}.$$

Back substitution results in $\mathbf{x} = (\frac{7}{5}, \frac{2}{5}, \frac{4}{5}, \frac{1}{5})^T$.

(c) (*Unseen*)

The trick is to see that $B = PAP$ where

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

is such that $P^{-1} = P$. So $B\mathbf{z} = \mathbf{b}$ is $A(P\mathbf{z}) = P\mathbf{b} = \mathbf{b}$ which means $P\mathbf{z} = \mathbf{x}$ and so $\mathbf{z} = P\mathbf{x}$. Only now do we need the calculation of $\mathbf{x}$ from part (b). So $\mathbf{z} = (\frac{1}{5}, \frac{4}{5}, \frac{2}{5}, \frac{7}{5})^T$.

2. (a) The function $g(x) = 2^{-x} = \exp(-x \ln 2)$ has values $g(1/3) = 0.7937$ and $g(1) = 0.5$ at the end points of the interval $[1/3, 1]$. It is a monotonously decreasing function and we conclude that $g(x) \in [1/3, 1]$ if $x \in [1/3, 1]$. In addition, $g(x)$ is differentiable and its derivative $g'(x) = -\exp(-x \ln 2) \ln 2$ has values $g'(1/3) = -0.5502$ and $g'(1) = -0.3466$ at the end points of the interval. The derivative $g'(x)$ is also monotonous and hence

$$|g'(x)| \leq |g'(1/3)| < 1 \qquad \text{if} \qquad x \in [1/3, 1]\,. \tag{1}$$

All conditions of the fixed-point theorem are satisfied and we conclude that there is a unique fixed point x^* in the interval $[1/3, 1]$, and that the fixed point iteration $x_{n+1} = g(x_n)$ converges to x^* for any initial point $x_0 \in [1/3, 1]$.

(b) We apply the mean value theorem to obtain an upper bound for the number of iterations that are required to determine the fixed point to an accuracy of 10^{-4}:

$$|x_n - x^*| = |g(x_{n-1}) - g(x^*)| = |g'(\xi_{n-1})|\,|x_{n-1} - x^*|\,, \tag{2}$$

where ξ_{n-1} is between x_{n-1} and x^* (and depends on x_{n-1}). We know that $|g'(\xi_{n-1})| \leq |g'(1/3)|$ because of inequality (1), and by using relation (2) iteratively we find

$$|x_n - x^*| \leq |g'(1/3)|^n\,|x_0 - x^*| \leq \frac{2}{3}|g'(1/3)|^n\,.$$

This error has to be smaller than 10^{-4}, and we obtain the following condition for n

$$\frac{2}{3}|g'(1/3)|^n \leq 10^{-4} \qquad \Longrightarrow \qquad n > [-4\ln(10) - \ln(2/3)]/\ln|g'(1/3)| \approx 14.74\,.$$

We conclude that 15 iterations are an upper bound.

3. (a) E.g. use `plot (x,coth(x))` in `wolframalpha.com` Since $\coth(x) > 1$ for $x > 0$, the root must lie to the right of $x = 1$.

(b) We need to show that the conditions of the fixed point theorem are satisfied. First, we need to show that for $x \in (1, 2)$, then $g(x) = \coth(x) \in [1, 2]$ also. We note that $g(x)$ is monotonically decreasing and is bounded below by 1. So we are only concerned with $g(1) = \coth(1) \approx 1.31 < 2$. I.e. $g(x) \in (1, 2)$. Second, we need to check that $|g'(x)| < 1$ for $x \in (1, 2)$. Here $g'(x) = 1/\sinh^2(x)$. This is monotonically decreasing and its largest value is at $x = 1$ where $g'(1) = 1/\sinh^2(1) \approx 0.74$. And so we are done: the FPT guarantees that an initial point $x_0 \in (1, 2)$ will converge to a unique $x^* \in (1, 2)$.

(c) The order of convergence is 1 (linear) since $g'(x) = 1/\sinh^2(x) \neq 0$ and so certainly not zero at $x = x^*$. The asymptotic error constant for linear convergence is (from notes) $|g'(x^*)|/1!$ which takes max/min values of 0.74, 0.076 in interval and so these are upper/ower bounds on the asymptotic error constant.

(d) The last column provides us with an estimate of the asymptotic error constant since we do not have the exact root x^*.

n	x_0	$\lvert x_n - x_{n-1}\rvert / \lvert x_{n-1} - x_{n-2}\rvert$
0	1.5	-
1	1.1047	-
2	1.2465	0.3587
3	1.1802	0.4675
4	1.2084	0.4253
5	1.1958	0.4468

(e) Newton is quadratic (in general) so let $f(x) = x \tanh x - 1$ be such that $f(x^*) = 0$. Now Newton is

$$x_{n+1} = x_n - \frac{x_n \tanh x_n - 1}{\tanh x_n + x_n \mathrm{sech}^2 x_n}.$$

4. (a) The first iteration scheme uses the function $g_1(x) = 20x/21 + 1/x^2$. We have

$$g_1(x) - x = \frac{21 - x^3}{21x^2} = 0\,.$$

This shows that $x^* = 21^{1/3}$ is the only fixed point. The convergence properties are investigated by evaluating the derivative of $g_1(x)$ at the fixed point.

$$g_1'(x^*) = \frac{20(x^*)^3 - 42}{21(x^*)^3} = \frac{18}{21}\,.$$

This is a constant between 0 and 1 and hence the convergence is linear.

(b) Now $g_2(x) = x - (x^3 - 21)/(3x^2)$ and the fixed point equation is

$$g_2(x) - x = -\frac{x^3 - 21}{3x^2} = 0\,.$$

As before $x^* = 21^{1/3}$ is the only fixed point. We evaluate the derivative of $g_2(x)$ at x^*

$$g_2'(x^*) = \frac{2(x^*)^3 - 42}{3(x^*)^3} = 0\,.$$

This shows that the convergence is faster than linear. To find the order of convergence we need to evaluate also the second derivative

$$g_2''(x^*) = \frac{42}{(x^*)^4} \neq 0\,.$$

The convergence is quadratic since the first derivative vanishes at the fixed point and the second derivative does not vanish at the fixed point.

(c) The third iteration scheme uses $g_3(x) = (21/x)^{1/2}$, and the fixed point equation can be written in the form

$$g_3(x) - x = \frac{21^{1/2} - x^{3/2}}{x^{1/2}} = 0\,.$$

Also in this case there is only one solution $x^* = 21^{1/3}$. We consider the derivative

$$g_3'(x^*) = -\frac{21^{1/2}}{2(x^*)^{3/2}} = -\frac{1}{2}.$$

We find that the convergence is linear, but quicker than for $g_1(x)$ since the derivative at x^* has a smaller modulus.

(d) Finally, $g_4(x) = x - (x^4 - 21x)/(x^2 - 21)$ and the fixed point equation takes the form

$$g_4(x) - x = -\frac{x(x^3 - 21)}{x^2 - 21} = 0.$$

Now we have two fixed points, $x_1^* = 21^{1/3}$ and $x_2^* = 0$. Starting with $x_0 = 1$ we find that the next value in the iteration scheme is $x_1 = 0$, and from then on $x_n = 0$ for $n \geq 1$. This shows that the iteration scheme converges to the fixed point $x_2^* = 0$ if $x_0 = 1$ and not to $x_1^* = 21^{1/3}$.

We conclude that only the first three iteration schemes converge to $21^{1/3}$. The quickest scheme uses $g_2(x)$, and the iteration with $g_3(x)$ is quicker than that with $g_1(x)$.

5. (a) The Newton scheme is

$$x_{n+1} = x_n - \tanh x_n / \mathrm{sech}^2 x_n \equiv x_n - \sinh x_n \cosh x_n \equiv x_n - \tfrac{1}{2}\sinh 2x_n$$

The results of iterating are

n	x_n	$\lvert x_n\rvert/\lvert x_{n-1}\rvert^3$
0	0.5	-
1	-8.760×10^{-2}	-0.7008
2	4.488×10^{-4}	-0.6676
3	-6.028×10^{-11}	-0.6666

(b) We set $g(x) = x - \frac{1}{2}\sinh 2x$, which is the RHS of the Newton iterative step. Then

$$g'(x) = 1 - \cosh 2x$$

and $g'(0) = 0$ as we expect from Newton. Then $g''(x) = -2\sinh 2x$ so that $g''(0) = 0$ also. Moving on, $g'''(x) = -4\cosh 2x$ so that $g'''(0) = -4$. Therefore we expect the scheme to be cubically convergent ($\alpha = 3$) and the asymptotic error constant, $\lambda = g'''(0)/3! = -2/3$. These calculations match the tabulated results since $x^* = 0$ so the 3rd column represents ratio of error at nth step to error at previous step raised to the power 3. The ratio tends towards $2/3$.

(c) We cannot use the section of the notes which estimates the interval of convergence directly because this assumes quadratic convergence and here we have cubic convergence. But the same approach applies and, by Taylors theorem we have

$$|x_{n+1} - x^*| = \frac{|g'''(\xi_n)|}{3!}|x_n - x^*|^3$$

for some ξ_n between x^* and x_n. We want the error at the $(n+1)$th step to be less that at the nth step and so

$$\frac{|g'''(\xi_0)|}{6}|x_0 - x^*|^2 < 1$$

is required at the first step. In our case $g'''(\xi_0) = -4\cosh 2\xi_0$ and $x^* = 0$. So we want

$$|x_0| < \sqrt{6/4\cosh\xi_0}$$

for some $\xi_0 \in (-x_0, x_0)$. The maximum value of $\cosh\xi_0$ in this interval is $\cosh x_0$, so we conclude that

$$|x_0| < \sqrt{6/4\cosh x_0}$$

is sufficient to ensure convergence.

(d) The first Newton step is $x_1 = x_0 - \frac{1}{2}\sinh 2x_0$. Assuming $x_0 > 0$ we want $-x_0 < x_1 < x_0$ to ensure we are closer to the root after the first step. This means $-x_0 < x_0 - \frac{1}{2}\sinh 2x_0$ and gives $4x_0 > \sinh 2x_0$. Not asked for, but this puts $|x_0| < 1.088$ (roughly).

With $x_0 = 1.5$ we find $x_1 = -3.509$, $x_2 = 275.6$ and this is clearly diverging.

6. The zero of $f(x) = (4x-7)/(x-2)$ is at $x^* = 7/4 = 1.75$. In the graphical interpretation of the Newton-Raphson method one draws a tangent to the graph of the function $f(x)$ at position $x = x_n$. The point where this tangent intersects the x-axis is the next iteration point x_{n+1}. The following figure shows a plot of the function $f(x)$. It has a pole at $x = 2$.

We can see from the plot that the Newton iteration scheme diverges if the initial point $x_0 > 2$, because then the next iteration points are at larger and larger values of x. This is illustrated by the tangent to the graph at position $x = 2.5$. The iteration scheme also diverges if the value of x_0 is so small that the next iteration point satisfies $x_1 \geq 2$. To find the border of this region we determine the value of x_0 for which the next iteration point would be $x_1 = 2$

$$2 = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad \Longrightarrow \qquad 0 = (x_0 - 2)(4x_0 - 6)\,.$$

The value $x_0 = 2$ is at the pole and the tangent at $x_0 = 1.5$ is shown in the figure. One can see that it intersects the x-axis at $x = 2$.

We conclude that the Newton iteration method diverges if $x_0 \leq 1.5$ or $x_0 \geq 2$. On the other hand, if $x_0 \in (1.5, 2)$ then one can convince oneself from the figure that all further iteration points also lie inside this interval and we would expect that the iteration scheme converges. Hence we expect convergence in the cases i) 1.625 ii) 1.875 and iv) 1.95, and divergence in the cases iii) 1.5 and v) 3. This is indeed the correct answer.

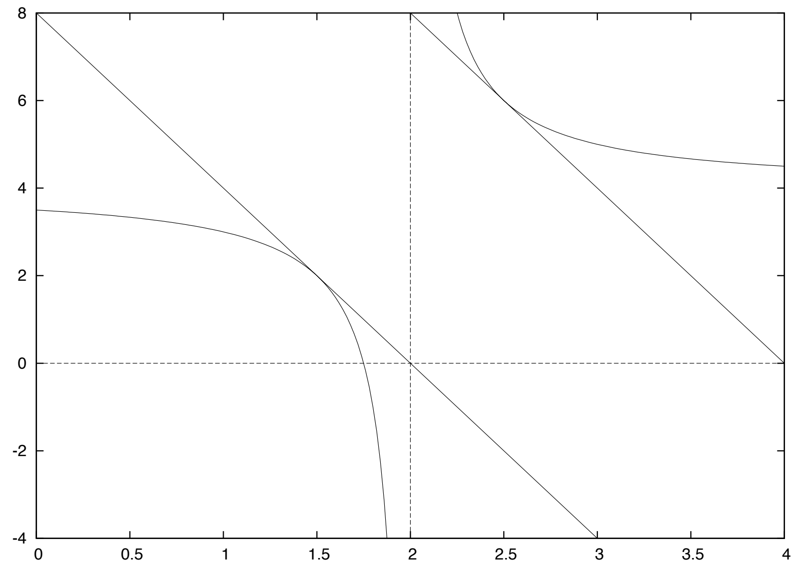

Figure 1: A plot of the function $f(x) = (4x-7)/(x-2)$. Two tangents are included, one at $x = 1.5$ and one at $x = 2.5$. The dotted vertical line denotes the pole at $x = 2$.

7. We consider the case that $f(x)$ has a simple zero at $x = x^*$ such that $f(x) = (x - x^*)\,q(x)$ where $q(x^*) \neq 0$. Then $f'(x^*) \neq 0$. The iteration scheme has the form of a fixed point iteration with

$$g(x) = x - \frac{f(x)}{f'(x)} - \frac{f''(x)\,f^2(x)}{2\,f'^3(x)}\,.$$

To find the order of convergence we evaluate the derivative of $g(x)$

$$g' = 1 - 1 + \frac{f\,f''}{f'^2} - \frac{f^2\,f''' + 2f\,f'\,f''}{2\,f'^3} + \frac{3\,f^2\,f''^2}{2\,f'^4} = f^2\,\frac{3\,f''^2 - f'\,f'''}{2f'^4}\,.$$

We see that $g'(x)$ has a zero at $x = x^*$ whose multiplicity is at least 2 because of the $f^2(x)$ term. (The fraction multiplying f^2 is in general not zero at $x = x^*$, but it can be in special cases). Hence the next derivative $g''(x)$ also vanishes at $x = x^*$ and the order of convergence is at least cubic.

8. We apply one step of the Newton iteration scheme to the function $f(x) = x^2 - a$ and denote the initial guess by x_0

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^2 - a}{2x_0} = \frac{1}{2}\left(x_0 + \frac{a}{x_0}\right)\,.$$

This is the required formula. To find the nth root of a number a we consider $f(x) = x^n - a$ and obtain

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0^n - a}{n\,x_0^{n-1}} = \frac{n-1}{n}x_0 + \frac{a}{n\,x_0^{n-1}}\,.$$

We apply this formula to estimate $\sqrt[3]{9}$ with an initial guess of $x_0 = 2$ and find

$$\sqrt[3]{9} \approx \frac{4}{3} + \frac{9}{12} = \frac{25}{12} = 2.083333\,.$$

The actual value is $\sqrt[3]{9} = 2.08008$.

9. (a) Letting $x_n \to x^*$ it is easy to see $x^* = 0$ satisfies the relation. To determine the order of convergence we let $g(x) = (2 - \sqrt{4 - x^2})^{1/2}$ and then

$$g'(x) = \frac{x/2}{\sqrt{4 - x^2}(2 - \sqrt{4 - x^2})^{1/2}}$$

and we should take the limit as $x \to 0^+$ to determine $g'(0)$. This gives

$$g'(0) = \lim_{x \to 0^+} \frac{x/2}{2(2 - 2(1 - (x/2)^2)^{1/2})^{1/2}}$$

after retaining only the terms that count at the end. Then using binomial expansions we get

$$g'(0) = \lim_{x \to 0^+} \frac{x/2}{2(x/2)} = 1/2$$

So the order of convergence is linear and the asymptotic error constant is 1/2. That is, we can anticipate that $x_n \approx x_{n-1}/2$ as $n \to \infty$.

(b) If we multiply both top and bottom by $\sqrt{2 + \sqrt{4 - x_n^2}}$ we are there after putting the numerator under a single square root and multiplying out factors. Simples.

(c) Here's the results:

	method 1		method 2	
n	x_n	$2^{n+1}x_n$	x_n	$2^{n+1}x_n$
1	0.7649	3.059	0.7652	3.061
2	0.3899	3.119	0.3900	3.120
3	0.1949	3.118	0.1960	3.136
4	0.1000	3.200	0.09815	3.141

Table 1: These numbers were produced by my daughter, Hazel, on her calculator ! She retained 4 significant figures after each step on her calculator.

(d) In both schemes $x_n \sim \pi/2^{n+1} \to 0$. In the second scheme we are dividing a small number by approximately 2 at each step and this is robust. In the first scheme, the final step of the calculation is roughly $\sqrt{(x_n/2)^2}$ which is prone to loss of accuracy due to round off errors.