Numerical Analysis **Solutions for sheet 7**

Orthogonal polynomials and Guassian quadrature

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1. (a) Draw the two graphs, spot there is only one intersection, at $x = x^*$, say, which is obviously positive and less than 1. Solutions of $x^3 + x - 1 = 0$ are equivalent to solutions of $x = 1/(1 + x^2)$ (since $1 + x^2$ is non-vanishing) and we are done.

   (b) The map is defined by $x_{n+1} = g(x_n)$ where $g(x) = 1/(1 + x^2)$.

   For $0 \le x \le 1$, $g(x) \in [\frac{1}{2}, 1] \subset [0, 1]$ since it is monotonically decreasing and takes its max/min values at $x = 0$, $x = 1$. Also

   $$|g'(x)| = \left| \frac{-2x}{(1 + x^2)^2} \right| < 1, \qquad \text{for } 0 \le x \le 1$$

   which requires some work to establish. For example,

   $$g''(x) = \frac{6x^2 - 2}{(1 + x^2)^3}$$

   implies there is a max/min in the interval $0 < x < 1$ at $x = 1/\sqrt{3}$ at which $|g'(1/\sqrt{3})| = 9/(8\sqrt{3}) < 1$. This is a maximum since $g'(0) = 0$ and $g'(1) = \frac{1}{2}$.

   Hence, by the Fixed Point Theorem, there exists a unique fixed point $x^* \in (0, 1)$ s.t. all $x_0 \in [0, 1]$ will converge to x^*.

   Finally, since $g'(x) \ne 0$ for $x \ne 0$ then $g'(x^*) \ne 0$ and so the scheme has first order convergence.

   (c) Here we are presented with $x_{n+1} = g(x_n)$ with

   $$g(x) = \frac{1 - x}{x^2}.$$

   Assuming a fixed point $x^* = (1 - x^*)/(x^*)^2$ rearranges to the original cubic. So same fixed point. Now

   $$g'(x) = \frac{-2 + x}{x^3}$$

   whose size is greater than 1 for all $0 < x < 1$. Hence $|g'(x^*)| > 1$ and the scheme cannot converge to x^* apart from if $x_0 = x^*$.

   (d) Here we are presented with $x_{n+1} = g(x_n)$ with

   $$g(x) = \frac{2x^3 + 1}{3x^2 + 1}.$$

   Assuming a fixed point $x^*(3(x^*)^2 + 1) = 2(x^*)^3 + 1$ rearranges to the original cubic. So same fixed point. Now

   $$g'(x) = \frac{6x^2(3x^2 + 1) - 6x(2x^3 + 1)}{(3x^2 + 1)^2} = \frac{6x(x^3 + x - 1)}{(3x^2 + 1)^2}$$

and so $g'(x^*) = 0$. Since $g(x)$ is continuous, there is a non-vanishing region around $x = x^*$ where $|g'(x)| < 1$ and this means the scheme will converge for x_0 sufficiently close to x^*.

(e) Different ways of getting to the required answer, but most direct is to set $f(x^*) = 0$ and write what is left in quadratic form

$$(x^* - x_n)^2 f''(x_n)/f'(x_n) + 2(x^* - x_n) + f(x_n)/f'(x_n) \approx 0$$

Solving gives

$$x^* - x_n \approx \frac{-1 \pm \sqrt{1 - 2f(x_n)f''(x_n)/[f'(x_n)]^2}}{f''(x_n)/f'(x_n)}$$

provided, obviously, that the square root is real. The issue here is which sign to take. We can assume that since x_n is close to x^* that $f(x_n)$ is small and use this to expand the square root using binomial expansion to

$$\sqrt{1 - 2f(x_n)f''(x_n)/[f'(x_n)]^2} \approx 1 - f(x_n)f''(x_n)/[f'(x_n)]^2$$

We see that if we take the $-$ root we end up with an equation which doesn't make much sense, but if we take the $+$ root we have

$$x^* - x_n \approx \frac{f'(x_n)}{f''(x_n)}\left(-1 + 1 - \frac{f(x_n)f''(x_n)}{[f'(x_n)]^2}\right) \approx -\frac{f(x_n)}{f'(x_n)}$$

which is Newton's method. So this tells us to choose the $+$ root. To form an iterative method, we let the approximation to x^* be x_{n+1} so that

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n)}\left(1 - \sqrt{1 - 2\frac{f(x_n)f''(x_n)}{[f'(x_n)]^2}}\right).$$

2. (a) (i) We have, with reference to the formula in the notes

$$P_3(x) = A_3\left(x\left(\frac{3}{2}x^2 - \frac{1}{2}\right) - \frac{\int_{-1}^{1} x(\frac{3}{2}x^2 - \frac{1}{2})^2\,dx}{\int_{-1}^{1}(\frac{3}{2}x^2 - \frac{1}{2})^2\,dx}\left(\frac{3}{2}x^2 - \frac{1}{2}\right) - \frac{\int_{-1}^{1} x^2(\frac{3}{2}x^2 - \frac{1}{2})\,dx}{\int_{-1}^{1} x^2\,dx}x\right)$$

and the middle term has an odd integrand in the integral in the numerator and therefore evaluates to zero. We have to be really careful with algebra here

$$\begin{aligned} P_3(x) &= A_3\left(\frac{3}{2}x^3 - \frac{1}{2}x - \left(\frac{\int_{-1}^{1}\frac{3}{2}x^4 - \frac{1}{2}x^2\,dx}{\frac{2}{3}}\right)x\right) \\ &= A_3\left(\frac{3}{2}x^3 - \frac{1}{2}x - \frac{3}{2}\left(\frac{3}{5} - \frac{1}{3}\right)x\right) \\ &= A_3\left(\frac{3}{2}x^3 - \frac{1}{2}x - \frac{3}{2}\left(\frac{4}{15}\right)x\right) = A_3\left(\frac{3}{2}x^3 - \frac{9}{10}x\right) \end{aligned}$$

Choosing $P_3(1) = 1$ implies $A_3 = \frac{5}{3}$ and so $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$.

(ii) Now $P_4(x)$ is trickier still

$$P_4(x) = A_3\left(x\left(\frac{5}{2}x^3 - \frac{3}{2}x\right) - \frac{\int_{-1}^{1} x(\frac{5}{2}x^3 - \frac{3}{2}x)(\frac{3}{2}x^2 - \frac{1}{2})\,dx}{\int_{-1}^{1}(\frac{3}{2}x^2 - \frac{1}{2})^2\,dx}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\right)$$

and the other term is suppressed because it involves the integral of an odd integrand. So

$$\begin{aligned}
P_4(x) &= A_4\left(\frac{5}{2}x^4 - \frac{3}{2}x^2 - \left(\frac{\int_0^1 15x^6 - 14x^4 + 3x^2\,dx}{\int_0^1 9x^4 - 6x^2 + 1}\right)\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\right) \\
&= A_4\left(\frac{5}{2}x^4 - \frac{3}{2}x^2 - \left(\frac{\frac{15}{7} - \frac{14}{5} + 1}{\frac{9}{5} - 2 + 1}\right)\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\right) \\
&= A_4\left(\frac{5}{2}x^4 - \frac{3}{2}x^2 - \frac{3}{7}\left(\frac{3}{2}x^2 - \frac{1}{2}\right)\right) = A_4\left(\frac{5}{2}x^4 - \frac{30}{14}x^2 + \frac{3}{14}\right).
\end{aligned}$$

Choosing $P_4(1) = 1$ implies $A_4 = \frac{7}{4}$ and so we get the answer required. Phew.

You wouldn't want to have to do $P_5(x)$. Luckily you don't because there's a recurrence relation which you can use to generate all these functions.

(b) The zeros of $P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$ are when $x^2 = 3$ or $x_1 = -\frac{1}{\sqrt{3}}$ and $x_2 = \frac{1}{\sqrt{3}}$. The weights are

$$w_1 = \int_{-1}^{1} w(x)\frac{(x - x_2)}{(x_1 - x_2)}\,dx = \frac{1}{2/\sqrt{3}}\int_{-1}^{1}(x + 1/\sqrt{3})\,dx = 1$$

since the integral of x is zero. Similarly,

$$w_2 = \int_{-1}^{1} w(x)\frac{(x - x_1)}{(x_2 - x_1)}\,dx = \frac{1}{-2/\sqrt{3}}\int_{-1}^{1}(x - 1/\sqrt{3})\,dx = 1.$$

Now

$$I = \int_0^1 \frac{1}{1 + x^2}\,dx = \frac{1}{2}\int_{-1}^{1}\frac{1}{1 + ((1 + t)/2)^2}\,dt$$

after the change of variables $x = (1 + t)/2$. So the $n = 2$ Gauss-Legendre quadrature applied to the approximation of the integral gives

$$I \approx \frac{1}{2}w_1\frac{1}{1 + ((1 + x_1)/2)^2} + \frac{1}{2}w_2\frac{1}{1 + ((1 + x_2)/2)^2}.$$

Or

$$I = \frac{2}{4 + (1 + 1/\sqrt{3})^2} + \frac{2}{4 + (1 - 1/\sqrt{3})^2} = \frac{2}{16/3 + 2/\sqrt{3}} + \frac{2}{16/3 - 2/\sqrt{3}} = \frac{48}{61} \approx 0.7868$$

The exact value of I is $[\tan^{-1}(x)]_0^1 = \pi/4 \approx 0.7853$.

(c) The answer is relatively obvious, which is that the integration interval is larger. The approximation to I is $f(-1/\sqrt{3}) + f(1/\sqrt{3})$ (2 point Gauss-Legendre quadrature) where $f(x) = 1/(1 + x^2)$ and this gives $I = 3/4 = 0.75$.

3. (a) We start with $\phi_0(x) = 1$. Next $\phi_1(x) = A_1x + B_1$ and we want $\phi_1(1) = A_1 + B_1 = 1$. We also want

$$0 = \int_0^1 x\phi_0(x)\phi_1(x)\,dx = \int_0^1 A_1x^2 + B_1x\,dx = \frac{1}{3}A_1 + \frac{1}{2}B_1.$$

Solving for A_1 and B_1 gives $\phi_1(x) = 3x - 2$.

Could use Gram-Schmidt, but let's do this one by hand. Let $\phi_2(x) = A_2x^2+B_2x+C_2$. Then $\phi_2(1) = A_2 + B_2 + C_2 = 1$. Also

$$0 = \int_0^1 x\phi_0(x)\phi_2(x)\,dx = \int_0^1 A_2x^3 + B_2x^2 + C_2x\,dx = \frac{1}{4}A_2 + \frac{1}{3}B_2 + \frac{1}{2}C_2.$$

Already we can eliminate, say, C_2 to leave $\frac{1}{2}A_1 + \frac{1}{3}B_1 = 1$. We finally need

$$0 = \int_0^1 x\phi_1(x)\phi_2(x)\,dx = \int_0^1 (3x^2 - 2x)(A_2x^2 + B_2x + C_2)\,dx = \frac{1}{4}A_2 + \frac{1}{3}B_2 + \frac{1}{2}C_2.$$

This simplifies to

$$0 = \frac{1}{10}A_2 + \frac{1}{12}B_2$$

and so $B_2 = -\frac{6}{5}A_2$ which combines with the condition already linking A_2 and B_2 to give $A_2 = 10$, $B_2 = -12$ and $C_2 = 3$. Hence $\phi_2(x)$ as given in question.

You can do this using Gram-Schmidt if you prefer (it's slightly more fiddly).

(b) We use the orthogonality property (see lectures) and multiply both sides by $w(x)\phi_0(x)$ and integrate over $0 < x < 1$ to give

$$\int_0^1 x^3\phi_0(x)\,dx = \sum_{n=0}^{2} a_n \int_0^1 x\phi_n(x)\phi_0(x)\,dx = a_0 \int_0^1 x\phi_0^2(x)\,dx$$

using orthogonality. Of course, $\phi_0(x) = 1$ so we get

$$a_0 = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}.$$

(c) Being asked for a 2-point approximation so need the two roots of $\phi_2(x) = 0$. These are when

$$x = \frac{12 \pm \sqrt{144 - 120}}{20} = \frac{6 \pm \sqrt{6}}{10}.$$

So let $x_1 = (6 - \sqrt{6})/10$ and $x_2 = (6 + \sqrt{6})/10$. Then

$$w_1 = \int_0^1 x\frac{(x - (6 + \sqrt{6})/10)}{((6 - \sqrt{6})/10 - (6 + \sqrt{6})/10)} = -\frac{10}{2\sqrt{6}}\left(\frac{1}{3} - \frac{1}{2}.\frac{(6 + \sqrt{6})}{10}\right) = \frac{1}{4} - \frac{\sqrt{6}}{36}$$

after tidying up the algebra. We do the same for w_2 with $-\sqrt{6}$ replaced by $+\sqrt{6}$ giving

$$w_2 = \frac{1}{4} + \frac{\sqrt{6}}{36}.$$

So together our 2-point approximation is

$$\int_0^1 xf(x)\,dx \approx \left(\frac{1}{4} - \frac{\sqrt{6}}{36}\right) f((6 - \sqrt{6})/10) + \left(\frac{1}{4} + \frac{\sqrt{6}}{36}\right) f((6 + \sqrt{6})/10).$$

(d) We know that the scheme in (c) exactly integrates cubic polynomials: that is if $f(x) = b_0 + b_1x + b_2x^2 + b_3x^3$ then

$$\int_0^1 x(b_0 + b_1x + b_2x^2 + b_3x^3)\,dx = \sum_{i=1}^{2} w_i(b_0 + b_1x_i + b_2x_i^2 + b_3x_i^3)$$

is EXACT. If we make the substitution $x = 1/t$ in the integral then we get $dx = (-1/t^2)dt$ and

$$\int_1^\infty (b_0/t^3 + b_1/t^4 + b_2/t^4 + b_3/t^6)\,dt = \sum_{j=1}^{2} w_j(b_0 + b_1x_j + b_2x_j^2 + b_3x_j^3).$$

We can make the RHS equal

$$\sum_{j=1}^{2} v_i(b_0/t_j^3 + b_1/t_j^4 + b_2/t_j^5 + b_3/t_j^6)$$

if we let $t_j = 1/x_j$ and $v_j = w_jt_j^3 = w_j/x_j^3$ for $j = 1, 2$.

4. (a) (i) the first two integrands are odd functions of x and so they integrate to zero. Using the substitution suggested

$$\int_{-1}^1 \sqrt{1-x^2}\,dx = \int_0^\pi \sin^2\theta\,d\theta = \pi/2$$

and

$$\int_{-1}^1 x^2\sqrt{1-x^2}\,dx = \int_0^\pi \cos^2\theta\sin^2\theta\,d\theta = \frac{1}{4}\int_0^\pi \sin^2(2\theta)\,d\theta = \pi/8$$

(ii) So $U_0(x) = 1$ is polynomial of degree 0 satisfying the standardisation condition. Next let $U_1(x) = A_1x + B_1$. Then we require

$$\langle U_0, U_1\rangle = 0 = A_1\int_{-1}^1 x\sqrt{1-x^2}\,dx + B_1\int_{-1}^1 \sqrt{1-x^2}\,dx = B_1\pi/2$$

It follows that $B_1 = 0$ and $A_1 = 2$ to ensure $U_1(1) = 2$. Thus $U_1(x) = 2x$. Next let $U_2(x) = A_2x^2 + B_2x + C_2$. Then we require

$$\langle U_0, U_2\rangle = 0 = A_2(\pi/8) + 0.B_2 + C_2(\pi/2)$$

after using results from (i) and

$$\langle U_1, U_2\rangle = 0 = 0.A_2 + (\pi/8)B_2 + 0.C_2$$

meaning $B_2 = 0$ and so $C_2 = -A_2/4$. Then $U_2(x) = A_2(x^2 - 1/4)$ and $U_2(1) = 3$ means $A_2 = 4$. So finally we have $U_2(x) = 4x^2 - 1$.

(iii) We are directed to choose $n = 2$ and so x_j are zeros of $U_2(x)$. I.e. we solve $4x^2 - 1 = 0$ which gives $x_1 = -1/2$ and $x_2 = +1/2$ (say). Then

$$w_1 = \int_{-1}^1 \sqrt{1-x^2}\frac{x - x_2}{x_1 - x_2}\,dx = \pi/4.$$

after inserting the definitions and using results from (i). Likewise

$$w_2 = \int_{-1}^{1} \sqrt{1-x^2}\frac{x-x_1}{x_2-x_1}\,dx = \pi/4.$$

Thus, the 2 point quadrature scheme is

$$\int_{-1}^{1} f(x)\sqrt{1-x^2}\,dx \approx \frac{\pi}{4}(f(-1/2)+f(1/2))$$

and this is exact if $f(x)$ is a polynomial of degree 3 or less.

(iv) The exact value of the integral is easy to determine: $\pi/4 \approx 1.273$.

To apply scheme to integral given, we need to define

$$f(x) = \cos(\pi x/2)/\sqrt{1-x^2}$$

Then

$$\int_{-1}^{1} \cos(\pi x/2)\,dx \approx \frac{\pi}{4}\left(\frac{\cos(-\pi/4)}{\sqrt{1-1/4}} + \frac{\cos(+\pi/4)}{\sqrt{1-1/4}}\right) = \frac{\pi}{\sqrt{6}} \approx 1.283.$$

So not bad then...

(b) (i) Defining x_j, $j = 1, \ldots, n$ to satisfy $U_n(x_j) = 0$ we have

$$\sin[(n+1)\cos^{-1}(x_j)] = 0$$

and so

$$(n+1)\cos^{-1}(x_j) = j\pi$$

or

$$x_j = \cos(j\pi/(n+1)), \qquad j = 1, \ldots, n.$$

(ii) Start with LHS:

$$U_{n+1}(x) + U_{n-1}(x) = \frac{\sin[(n+1+1)\cos^{-1}(x)]}{\sin[\cos^{-1}(x)]} + \frac{\sin[(n-1+1)\cos^{-1}(x)]}{\sin[\cos^{-1}(x)]}$$

and the RHS is

$$\frac{2\sin[(n+1)\cos^{-1}(x)\cos[\cos^{-1}(x)]}{\sin[\cos^{-1}(x)]} = 2xU_n(x)$$

So here $f(x) = 2x$.

We can see directly from $n = 0$ that $U_0 = 1$ and from $n = 2$ that ($\sin 2\theta = 2\sin\theta\cos\theta$)

$$U_1(x) = \frac{\sin[2\cos^{-1}(x)]}{\sin[\cos^{-1}(x)]} = 2\cos[\cos^{-1}(x)] = 2x.$$

So the formula works for $n = 0$ and $n = 1$ whereby $U_n(x)$ is a polynomial of degree n. The recurrence relation shows, by induction, that if $u_n(x)$ is a polynomial of degree n, then $U_{n+1}(x)$ is a polynomial of degree $n+1$.

Finally, we have to be a bit clever about the standardisation condition being met as it's a 0/0 limit:

$$U_n(1) = \lim_{x\to 1} \frac{\sin[(n+1)\cos^{-1}(x)]}{\sin[\cos^{-1}(x)]} = \lim_{y\to 0} \frac{\sin[(n+1)y]}{\sin y} = (n+1)$$

after using $y = \cos^{-1}(x)$.

(iii) Follow methods in notes for Chebychev polynomials. Similar approach and it works.

5. We first calculate some integrals that we will need in the following. We use the substitution $x = \cos\theta$, $\mathrm{d}x = -\sin\theta\,\mathrm{d}\theta$ to find

$$\begin{aligned}
\int_{-1}^{1} w(x)\,\mathrm{d}x &= \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}dx = \int_0^\pi \frac{\sin\theta}{\sin\theta}\mathrm{d}\theta = \int_0^\pi \mathrm{d}\theta = \pi\,, \\
\int_{-1}^{1} w(x)\,x^{2n+1}\,\mathrm{d}x &= \int_{-1}^{1} \frac{x^{2n+1}}{\sqrt{1-x^2}}dx = 0\,, \qquad (1) \\
\int_{-1}^{1} w(x)\,x^2\,\mathrm{d}x &= \int_{-1}^{1} \frac{x^2}{\sqrt{1-x^2}}dx = \int_0^\pi \cos^2\theta\,\mathrm{d}\theta = \int_0^\pi \frac{1}{2}[1+\cos(2\theta)]\,\mathrm{d}\theta = \frac{\pi}{2}, \\
\int_{-1}^{1} w(x)\,x^4\,\mathrm{d}x &= \int_0^\pi \cos^4\theta\,\mathrm{d}\theta = \int_0^\pi \frac{1}{4}[1+\cos(2\theta)]^2\,\mathrm{d}\theta = \frac{3\pi}{8}.
\end{aligned}$$

The second integral vanishes because it is an integral over an odd function.

(a) We have $T_0(x) = 1$ and

$$T_1(x) = A_1\left(x - \frac{\langle x,1\rangle}{\langle 1,1\rangle}\right) = A_1 x = x$$

using (1) and standardisation condition. Next

$$T_2(x) = A_2\left(x^2 - \frac{\langle x^2,x\rangle}{\langle x,x\rangle}x - \frac{\langle x^2,1\rangle}{\langle 1,1\rangle}\right) = A_2\left(x^2 - 1/2\right)$$

using (1). Then $A_2 = 2$ so that $T_2(x) = 2x^2 - 1$. Next

$$T_3(x) = A_3\left(x(2x^2-1) - \frac{\langle x(2x^2-1),(2x^2-1)\rangle}{\langle (2x^2-1),(2x^2-1)\rangle}(2x^2-1) - \frac{\langle x(2x^2-1),x\rangle}{\langle x,x\rangle}x\right)$$

The middle term is zero since it involves odd powers of x. This leaves us with

$$T_3(x) = A_3\left(2x^3 - x - x\frac{(3\pi/4 - \pi/2)}{\pi/2}\right) = A_3\left(2x^3 - \frac{3}{2}x\right)$$

Then we find $A_3 = 2$ so that $T_3(1) = 1$ and $T_3(x) = 4x^3 - 3x$.

The relations to $T_n(x) = \cos[n\cos^{-1}(x)]$ follow since if we let $x = \cos\theta$, $T_2(x) = 2x^2 - 1$ translates to $\cos 2\theta = 2\cos^2\theta - 1$ and $T_3(x) = 4x^3 - 3x$ is just $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ which are both standard results.

(b) We can now calculate the weights in the Gauss-Chebyshev quadrature formula. For $n = 1$ the roots of $T_1(x)$ are $x_1 = 0$. We use equations (1) and the formula for the weight to obtain

$$w_1 = \int_{-1}^{1} w(x)\,\mathrm{d}x = \pi.$$

For $n = 2$ we have the roots of $T_2(x)$ are $x_1 = -1/\sqrt{2}$, $x_2 = 1/\sqrt{2}$. The formulae for the weights are

$$w_1 = \int_{-1}^{1} w(x)\left(\frac{x - x_2}{x_1 - x_2}\right)\mathrm{d}x = \pi\left(\frac{-x_2}{x_1 - x_2}\right) = \frac{\pi}{2},$$
$$w_2 = \int_{-1}^{1} w(x)\left(\frac{x - x_1}{x_2 - x_1}\right)\mathrm{d}x = \pi\left(\frac{-x_1}{x_2 - x_1}\right) = \frac{\pi}{2},$$

where we used equations (1).

For $n = 3$ the roots of $T_3(x)$ are $x_1 = -\sqrt{3}/2$, $x_2 = 0$, $x_3 = \sqrt{3}/2$ and we have

$$w_1 = \int_{-1}^{1} w(x)\left(\frac{x - x_2}{x_1 - x_2}\right)\left(\frac{x - x_3}{x_1 - x_3}\right)\mathrm{d}x = \frac{\pi + 2\pi\, x_2 x_3}{2\,(x_1 - x_2)\,(x_1 - x_3)} = \frac{\pi}{3},$$
$$w_2 = \int_{-1}^{1} w(x)\left(\frac{x - x_1}{x_2 - x_1}\right)\left(\frac{x - x_3}{x_2 - x_3}\right)\mathrm{d}x = \frac{\pi + 2\pi\, x_1 x_3}{2\,(x_2 - x_1)\,(x_2 - x_3)} = \frac{\pi}{3},$$
$$w_3 = \int_{-1}^{1} w(x)\left(\frac{x - x_1}{x_3 - x_1}\right)\left(\frac{x - x_2}{x_3 - x_2}\right)\mathrm{d}x = \frac{\pi + 2\pi\, x_1 x_2}{2\,(x_3 - x_1)\,(x_3 - x_2)} = \frac{\pi}{3},$$

where we used equations (1) again.

We see that in all cases the results agree with the general formula $w_j = \pi/n$, $j = 1, \ldots, n$.

(c) If you just use $f(x) = \ln|x - t|$ you can see there is a singularity at $x = t$ and $f(x)$ is not very much like a smooth polynomial. This will give poor results. So we have to think of ways to remove the singularity.

If $t \neq \pm 1$ we can propose the follow

$$\int_{-1}^{1} \frac{\ln|x - t|}{\sqrt{1 - x^2}} = \int_{-1}^{1} \frac{\ln|x - t|}{\sqrt{1 - x^2}} - \frac{\ln|x - t|}{\sqrt{1 - t^2}}\,dt + \frac{1}{\sqrt{1 - t^2}} \int_{-1}^{1} \ln|x - t|\,dx.$$

Then the final term can be integrated explicitly since

$$\begin{aligned}
\int_{-1}^{1} \ln|x - t|\,dx &= \int_{-1}^{t} \ln(t - x)\,dx + \int_{t}^{1} \ln(x - t)\,dx \\
&= [x - (t - x)\ln(t - x)]_{-1}^{t} + [(x - t)\ln(x - t) - x]_{t}^{1} \\
&= [(t + 1) + (t + 1)\ln(t + 1)] + [(1 - t) - (1 - t)\ln(1 - t)].
\end{aligned}$$

The integral on the RHS can be written

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}}\,dx$$

where

$$f(x) = \ln|x-t|\left(1 - \frac{\sqrt{1-x^2}}{\sqrt{1-t^2}}\right)$$

s.t. $f(x) \to 0$ as $x \to t$ and so the singularity has been removed. Indeed, $f(x)$ is a smooth function of x to which Gauss-Chebyshev quadrature can be used effectively. However, I don't yet have an answer for what to do when $t = \pm 1$.

(a) (a) When n is odd, x^n is an odd function and so multiplying by the even function e^{-x^2} and integrating over $-\infty < x < \infty$ is zero.

Note $I_0 = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ is a standard integral.

Now we integrate by parts

$$I_{2n} = \int_{-\infty}^{\infty} x^{2n-1}\left(xe^{-x^2}\right)\,dx = \left[-\frac{x^{2n-1}e^{-x^2}}{2}\right]_{-\infty}^{\infty} + \frac{2n-1}{2}\int_{-\infty}^{\infty} x^{2n-2}e^{-x^2}\,dx$$

so that

$$I_{2n} = \frac{(2n-1)}{2} I_{2n-2}$$

as required. Now apply repeatedly to get

$$I_{2n} = \frac{(2n-1)(2n-3)\ldots 3.1}{2^n} I_0$$

and recognise that

$$(2n-1)(2n-3)\ldots 3.1 = \frac{(2n)(2n-1)(2n-2)(2n-3)\ldots 3.2.1}{(2n)(2n-2)\ldots 4.2} = \frac{(2n)!}{2^n n!}$$

Then putting back together gives desired result.

(b) Let's do it using method 1 of the notes. So

$$H_0(1) = 1$$

satisfies the standardisation condition. Next, we choose $H_1(x) = x + B_1$, which satisfies the standardisation condition. And then

$$0 = \langle H_1, H_0\rangle = \langle x, 1\rangle + B_1\langle 1, 1\rangle = I_1 + B_1 I_0$$

so $B_1 = 0$ and $H_1(x) = x$. Next let $H_2 = x^2 + B_2 x + C_2$ which again satisfies the standardisation condition and now

$$0 = \langle H_2, H_0\rangle = \langle x^2, 1\rangle + B_2\langle x, 1\rangle + C_2\langle 1, 1\rangle = \frac{\sqrt{\pi}}{2} + C_2\sqrt{\pi}$$

gives $C_2 = -\frac{1}{2}$ and

$$0 = \langle H_2, H_1\rangle = \langle x^2, x\rangle + B_2\langle x, x\rangle + C_2\langle 1, x\rangle = B_2\frac{\sqrt{\pi}}{2}$$

so $B_2 = 0$. Thus

$$H_2(x) = x^2 - 1/2.$$

(c) Need 2-point Gaussian Quadrature. So the roots of $x^2 - 1/2 = 0$ are $x_1 = -1/\sqrt{2}$, $x_2 = +\frac{1}{\sqrt{2}}$. The weights are defined by the usual fomula

$$w_1 = \int_{-\infty}^{\infty} \frac{x - x_1}{x_2 - x_1} \mathrm{e}^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$$

and

$$w_2 = \int_{-\infty}^{\infty} \frac{x - x_2}{x_1 - x_2} \mathrm{e}^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

(d) Here we want to define $f(x) = \mathrm{e}^{x^2} 1 + x^2$ so that

$$I = \int_{-\infty}^{\infty} \frac{1}{1 + x^2}\, dx = \int_{-\infty}^{\infty} f(x) w(x)\, dx \approx \frac{\sqrt{\pi}}{2} \left(\frac{\mathrm{e}^{1/2}}{1 + \frac{1}{2}} + \frac{\mathrm{e}^{1/2}}{1 + \frac{1}{2}} \right) = \frac{2\sqrt{\pi}}{3} \mathrm{e}^{1/2}$$

This is 1.94818 and the exact answer is π which is not so good.

(e) A trickier question.