Methods for solving Initial Value Problems

---

1. (a) We find using integrating factors that

$$\frac{d}{dt}(y\mathrm{e}^{-t}) = -\mathrm{e}^{-2t}$$

and this integrates to

$$y\mathrm{e}^{-t} = \frac{1}{2}\mathrm{e}^{-2t} + C$$

where $C = \frac{1}{2}$ from the initial condition and so

$$y(t) = \frac{1}{2}\mathrm{e}^{t} + \frac{1}{2}\mathrm{e}^{-t} = \cosh(t).$$

(b) Euler's method with $t = ih$ and h is the step-size

$$y_{i+1} = y_i + h(y_i - \mathrm{e}^{-ih}), \qquad y_0 = 1$$

We solve by letting $y_i = y_i^h + y_i^p$, solutions to the homogeneous difference equation and a particular solution. Thus

$$y_{i+1}^h = y_i^h(1+h) \qquad \text{and} \qquad y_{i+1}^p = y_i^p(1+h) - h\mathrm{e}^{-ih}$$

We let $y_i^h = Ar^i$ and get

$$Ar^{i+1} = Ar^i(1+h)$$

so that $r = (1+h)$ and $y_i^h = A(1+h)^i$. For the particular solution, try $y_i^p = B\mathrm{e}^{-ih}$. Then

$$B\mathrm{e}^{-(i+1)h} - B(1+h)\mathrm{e}^{-ih} = -h\mathrm{e}^{-ih}$$

in which we can cancel e^{-ih} terms to leave

$$B = h/(1+h-\mathrm{e}^{-h})$$

So we now have $y_i^p = h\mathrm{e}^{-ih}/(1+h-\mathrm{e}^{-h})$ and a general solution of

$$y_i^p = \frac{A(1+h)^i + h\mathrm{e}^{-ih}}{(1+h-\mathrm{e}^{-h})}.$$

Finally, we impose $y_0 = 1$ which gives $A = 1 - \mathrm{e}^{-h}$ and that's the solution in the question.

(c) For the exact solution we have $y(ih) = \cosh(ih) \approx 1 + i^2h^2/2 + \ldots$ for $ih \ll 1$ (the MacLaurin series for cosh). We need a similar series expansion from the Euler solution in part (b) and expand using binomial and MacLaurin to give

$$y_i = \frac{(1-\mathrm{e}^{-h})(1 + ih + i(i-1)h^2/2 + \ldots) + h(1 - ih + i^2h^2/2 + \ldots)}{1+h-\mathrm{e}^{-h}}$$

and we can write this as

$$y_i = 1 + \frac{i^2h^2}{2} + \frac{(1-\mathrm{e}^{-h})(ih - ih^2/2) - ih^2 + \ldots}{1 + h - \mathrm{e}^{-h}}.$$

Now we expand

$$y_i = 1 + \frac{i^2h^2}{2} + \frac{(h - h^2/2 + \ldots)(ih - ih^2/2) - ih^2 + \ldots}{2h - h^2/2 + \ldots}$$

and simplify with a bit of binomial

$$y_i \approx 1 + \frac{i^2h^2}{2} + \frac{-ih^3 + \ldots}{2h}(1 + h/4 + \ldots)$$

and the error between exact and Euler is therefore

$$y(ih) - y_i = \frac{ih^2}{2} + O(h^3)$$

provided $ih \ll 1$. Thus pick up the local truncation error and agrees with the notes for Euler's method. To establish the error at $t = 1$, we set $i = N$ where $N = 1/h$ and now $ih = 1 \not\ll 1$. So we need a different approach. The exact solution is

$$y(1) = \frac{1}{2}\mathrm{e} + \frac{1}{2}\mathrm{e}^{-1}$$

For Euler we set $i = 1/h$ in the solution to the difference equation to get

$$y_N = \frac{(1-\mathrm{e}^{-h})(1+h)^{1/h} + h\mathrm{e}^{-h/h}}{1 + h - \mathrm{e}^{-h}}$$

and make expansions based on $h \ll 1$ so that

$$y_N = \frac{(h - h^2/2 + \ldots)\exp\{(1/h)\ln(1+h)\} + h\mathrm{e}^{-1}}{2h - h^2/2 + \ldots}$$

We need to be very very careful with our expansions to make sure we keep enough terms. So I get, after throwing away some higher-order terms which will not count

$$y_N \approx \frac{(1-h/2)}{2(1-h/4)}\exp\{(1/h)(h - h^2/2 + \ldots)\} + \frac{\mathrm{e}^{-1}}{2(1-h/4)}$$

after using expansion for the log. Next, expand further to get

$$y_N \approx \frac{1}{2}(1-h/2)(1+h/4)\mathrm{e}^{1-h/2+\cdots} + \frac{1}{2}\mathrm{e}^{-1}(1+h/4)$$

or

$$y_N \approx \frac{1}{2}\mathrm{e}(1-h/2)(1+h/4)(1-h/2) + \frac{1}{2}\mathrm{e}^{-1}(1+h/4).$$

So I'm pretty confident we have

$$y_N = \cosh(1)(1+h/4) - \frac{1}{2}h\mathrm{e} + O(h^2)$$

and hence the global error is $-h\cosh(1)/4 + h\mathrm{e}/2 = O(h)$.

2. This is from the notes. The Taylor expansion of $y(t_i + h)$ up to third order is

$$y(t_i + h) = y(t_i) + hy'(t_i) + \frac{h^2}{2} y''(t_i) + \frac{h^3}{6} y'''(t_i) + \mathcal{O}(h^4)\,. \tag{1}$$

The differential equation $y'(t) = f(t, y(t))$ is used to replace the derivatives of $y(t)$

$$\begin{aligned} y''(t) &= \frac{d}{dt} f(t, y) = f_t(t, y(t)) + f_y(t, y(t))\, y'(t) = f_t + f_y f\,, \\ y'''(t) &= \frac{d}{dt}[f_t + f_y\, f] = f_{tt} + f_{ty}\, f + f_{yt} f + f_{yy} f^2 + f_y f_t + f_y f_y f \end{aligned}$$

Neglecting the $\mathcal{O}(h^4)$ term in (1) one obtains the following iteration scheme

$$\begin{aligned} y_{i+1} = y_i + hf(t_i, y_i) + \frac{h^2}{2}[f_t(t_i, y_i) + f_y(t_i, y_i)\, f(t_i, y_i)] + \frac{h^3}{6}[f_{tt}(t_i, y_i) \\ + 2 f_{ty}(t_i, y_i)\, f(t_i, y_i) + f_{yy}(t_i, y_i)\, f^2(t_i, y_i) + f_y(t_i, y_i)\, f_t(t_i, y_i) + f_y^2(t_i, y_i)\, f(t_i, y_i)]\,. \end{aligned}$$

From notes with $f(t, y) = 2ty$ and $t = ih$:

(a) Euler is

$$y_{i+1} = y_i + 2ht_i y_i = y_i(1 + 2ih^2)$$

(b) Taylor's method of order two ($f_t = 2y,\ f_y = 2t$) is

$$y_{i+1} = y_i + 2ht_i y_i + \frac{h^2}{2}\left[2y_i + 4y_i t_i^2\right] = y_i(1 + 2ih^2 + h^2 + 2i^2h^4).$$

(c) Taylor's method of order three ($f_{tt} = f_{yy} = 0,\ f_{yt} = 2$) is what we got above plus the $O(h^3)$ term:

$$y_{i+1} = y_i(1 + 2ih^2 + h^2 + 2i^2h^4) + \frac{h^3}{6}\left[8t_i y_i + 4y_i t_i + 8y_i t_i^3\right]$$

and this comes out to be

$$y_{i+1} = y_i(1 + 2ih^2 + h^2 + 2i^2h^4 + 2ih^4 + (4/3)i^3h^6).$$

3. We start from the initial value problem

$$y''' = y' + y\,, \qquad t \in [0, 1]\,, \qquad y(0) = 0\,, \qquad y'(0) = 1\,, \qquad y''(0) = 0\,.$$

This third order ODE can be transformed into a system of first order ODEs by setting $u(t) = y'(t)$ and $v = y''(t)$. This yields

$$\begin{aligned} y' &= u\,, & y(0) &= 0\,, & t \in [0, 1]\,, \\ u' &= v\,, & u(0) &= 1\,, & \\ v' &= u + y\,, & v(0) &= 0\,. & \end{aligned}$$

We apply Euler's method to each of these lines and let y_i, u_i and v_i denote the approximations for $y(t_i)$, $u(t_i)$ and $v(t_i)$, respectively. This results in

$$
\begin{aligned}
y_{i+1} &= y_i + h\,u_i\,, & y_0 &= 0\,,\\
u_{i+1} &= u_i + h\,v_i\,, & u_0 &= 1\,,\\
v_{i+1} &= v_i + h\,(u_i + y_i)\,, & v_0 &= 0\,.
\end{aligned}
$$

This is a system of first order difference equations which can be solved by iteration. Alternatively, we can obtain from the second line $v_i = (u_{i+1} - u_i)/h$ and from the first line $u_i = (y_{i+1} - y_i)/h$, and we can use these relations to eliminate the u and v variables. This leads to a third order difference equation in the variable y.

$$
\frac{y_{i+3} - 2y_{i+2} + y_{i+1}}{h^2} = \frac{y_{i+2} - 2y_{i+1} + y_i}{h^2} + h\left(\frac{y_{i+1} - y_i}{h} + y_i\right)\,,
$$

with initial conditions

$$
y_0 = 0\,, \qquad \frac{y_1 - y_0}{h} = 1\,, \qquad \frac{y_2 - 2y_1 + y_0}{h^2} = 0\,.
$$

This can be rewritten in the form

$$
y_{i+3} = 3y_{i+2} - 3y_{i+1} + y_i + h^2\left(y_{i+1} - y_i + h\,y_i\right)\,,
$$

with initial conditions

$$
y_0 = 0\,, \qquad y_1 = h\,, \qquad y_2 = 2h\,.
$$

4. We apply Taylor's expansion in two variable to the equation

$$
y_{i+1} = y_i + af(t_i, y_i) + bf(t_i + c, y_i + d)\,.
$$

and obtain

$$
y_{i+1} = y_i + af(t_i, y_i) + bf(t_i, y_i) + bcf_t(t_i, y_i) + bdf_y(t_i, y_i) + \frac{b}{2}[f_{tt}c^2 + 2f_{ty}cd + f_{yy}d^2] + \ldots
$$

This is compared to Taylor's method of order 2

$$
y_{i+1} = y_i + hf(t_i, y_i) + \frac{h^2}{2}[f_t(t_i, y_i) + f_y(t_i, y_i)\,f(t_i, y_i)]
$$

We find that

$$
a + b = h, \quad bc = \frac{h^2}{2}, \quad bd = \frac{h^2}{2}f(t_i, y_i).
$$

These are three conditions for the four unknowns a, b, c and d. In the first case (modified Euler method) we require also $a = b$ and obtain

$$
a = b = \frac{h}{2}, \;\; c = h, \;\; d = hf(t_i, y_i), \qquad y_{i+1} = y_i + \frac{h}{2}f(t_i, y_i) + \frac{h}{2}f(t_i + h, y_i + hf(t_i, y_i))\,.
$$

In the second case we require $3a = b$ instead of $a = b$ and obtain

$$
a = \frac{h}{4}, \quad b = \frac{3h}{4}, \quad c = \frac{2h}{3}, \quad d = \frac{2h}{3}f(t_i, y_i),
$$

and hence

$$
y_{i+1} = y_i + \frac{h}{4}f(t_i, y_i) + \frac{3h}{4}f(t_i + \frac{2h}{3}, y_i + \frac{2h}{3}f(t_i, y_i)).
$$

	$h = 0.1$	$h = 0.05$	quotient
RK2	-0.00922482	-0.00229199	$4.02 \approx 4$
RK4	-0.00001164	-0.00000075	$15.5 \approx 16$

5. The exact solution is $y(t) = \exp(t^2)$ and $y(1) = \mathrm{e} \approx 2.71828183$. The table below shows the errors of the approximations. The last column contains the quotients of the errors for $h = 0.1$ and $h = 0.05$. The quotients agrees with order of accuracy 2 and 4, respectively, because halving the value of h leads to an approximate multiplication of the error by 2^{-2} and 2^{-4}, respectively.