Numerical Analysis **Solutions for sheet 9**

Multistep methods

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1. The local truncation error for Milne's implicit 2-step method

$$y_{i+1} = y_{i-1} + h\beta_0 f(t_{i+1}, y_{i+1}) + h\beta_1 f(t_i, y_i) + h\beta_2 f(t_{i-1}, y_{i-1}),$$

is obtained by using $f(t, y(t)) = y'(t)$ and Taylor expansion

$$\begin{aligned}
\tau_{i+1} &= y(t_i + h) - y(t_i - h) - h\beta_0 y'(t_i + h) - h\beta_1 y'(t_i) - h\beta_2 y'(t_i - h) \\
&= y + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' + \frac{h^4}{24}y^{(4)} + \frac{h^5}{120}y^{(5)} + \ldots - y + hy' - \frac{h^2}{2}y'' + \frac{h^3}{6}y''' - \frac{h^4}{24}y^{(4)} \\
&\quad + \frac{h^5}{120}y^{(5)} + \ldots - h\beta_0\left[y' + hy'' + \frac{h^2}{2}y''' + \frac{h^3}{6}y^{(4)} + \frac{h^4}{24}y^{(5)} + \ldots\right] - h\beta_1 y' \\
&\quad - h\beta_2\left[y' - hy'' + \frac{h^2}{2}y''' - \frac{h^3}{6}y^{(4)} + \frac{h^4}{24}y^{(5)} + \ldots\right] \\
&= hy'[2 - \beta_0 - \beta_1 - \beta_2] + h^2 y''[-\beta_0 + \beta_2] + \frac{h^3}{6}y'''[2 - 3\beta_0 - 3\beta_2] \\
&\quad + \frac{h^4}{6}y^{(4)}[-\beta_0 + \beta_2] + \frac{h^5}{120}y^{(5)}[2 - 5\beta_0 - 5\beta_2] + \ldots
\end{aligned}$$

We require that the coefficients of the first three derivatives of y vanish

$$\beta_0 + \beta_1 + \beta_2 = 2, \quad \beta_0 = \beta_2, \quad 3\beta_0 + 3\beta_2 = 2, \quad \Longrightarrow \quad \beta_0 = \beta_2 = \frac{1}{3}, \quad \beta_1 = \frac{4}{3}.$$

The coefficient of $y^{(4)}$ vanishes then also and the local truncation error is $\mathcal{O}(h^5)$. The corresponding order of accuracy is 4. The final result for the iteration scheme is

$$y_{i+1} = y_{i-1} + \frac{h}{3}\left[f(t_{i+1}, y_{i+1}) + 4f(t_i, y_i) + f(t_{i-1}, y_{i-1})\right].$$

2. The local truncation error is defined by (following method in notes)

$$\begin{aligned}
\tau_{i+1} &= y(t_i + h) - \alpha_1 y(t_i) - \alpha y(t_i - h) - h\beta_0 y'(t_i + h) \\
&= y(t_i) + hy'(t_i) + \tfrac{1}{2}h^2 y''(t_i) + \tfrac{1}{6}h^3 y'''(t_i) + \ldots \\
&\quad -\alpha_1 y(t_i) \\
&\quad -\alpha_2\left[y(t_i) - hy'(t_i) + \tfrac{1}{2}h^2 y''(t_i) - \tfrac{1}{6}h^3 y'''(t_i) + \ldots\right] \\
&\quad -h\beta_0\left[y'(t_i) + hy''(t_i) + \tfrac{1}{2}h^2 y'''(t_i) + \tfrac{1}{6} + \ldots\right]
\end{aligned}$$

We eliminate a series of terms... To get rid of terms proportional to $y(t_i)$, $y'(t_i)$, $y''(t_i)$ in turn gives the three relations

$$1 - \alpha_1 - \alpha_2, \qquad 1 + \alpha_2 - \beta_0, \qquad 1 - \alpha_2 - 2\beta_0$$

which when solved give $\beta_0 = 2/3$, $\alpha_1 = 4/3$ and $\alpha_2 = -1/3$. The remaining term proportional to $y'''(t_i)$ does not vanish and is $O(h^3)$. Therefore the local truncation error is $O(h^3)$ and the order of accuracy is 2.

3. The convergence of a multistep formula is proved by showing two points. First, that the local truncation error is of order $\mathcal{O}(h^{p+1})$ with $p > 0$, and second, that the formula is stable. If these two conditions are satisfied then it follows from Dahlquist's theorem that the global error is $\mathcal{O}(h^p)$, and the formula converges to the exact solution as h goes to zero if $p > 0$.

   The stability of a multistep formula follows from the root condition: the formula is stable if and only if all roots of the characteristic polynomial satisfy $|z| \leq 1$ and any root with $|z| = 1$ is simple. The characteristic polynomial is obtained by setting $f_i = 0$ and $y_i = z^i$.

   (a) $y_{i+1} = y_i$. Local truncation error:

   $$\tau_{i+1} = y(t_i + h) - y(t_i) = y + hy' - y + \mathcal{O}(h^2) = hy' + \dots.$$

   The order of accuracy is $p = 0$ and the formula is not convergent. (Note that it does not contain any information about the differential equation.) The stability polynomial is $z - 1 = 0$. It satisfies the root condition and the formula is stable.

   (b) $y_{i+1} = y_{i-3} + \frac{4}{3}h(f_i + f_{i-1} + f_{i-2})$. Local truncation error:

   $$\begin{aligned}\tau_{i+1} &= y(t_i + h) - y(t_i - 3h) - \frac{4}{3}h(y'(t_i) + y'(t_i - h) + y'(t_i - 2h)) \\ &= y + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' - y + 3hy' - \frac{9h^2}{2}y'' + \frac{27h^3}{6}y''' \\ &\quad - \frac{4h}{3}\left(y' + y' - hy'' + \frac{h^2}{2}y''' + y' - 2hy'' + \frac{4h^2}{2}y'''\right) + \mathcal{O}(h^4) \\ &= \frac{4}{3}h^3y''' + \dots\end{aligned}$$

   The stability polynomial is $0 = z^4 - 1 = (z-1)(z+1)(z-i)(z+i)$. The root condition is satisfied, the order of accuracy is $p = 2$, and the formula is convergent.

   (c) $y_{i+1} = y_{i-1} + \frac{1}{3}h(7f_i - 2f_{i-1} + f_{i-2})$. Local truncation error:

   $$\begin{aligned}\tau_{i+1} &= y(t_i + h) - y(t_i - h) - \frac{h}{3}(7y'(t_i) - 2y'(t_i - h) + y'(t_i - 2h)) \\ &= y + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' + \frac{h^4}{24}y^{(4)} - y + hy' - \frac{h^2}{2}y'' + \frac{h^3}{6}y''' - \frac{h^4}{24}y^{(4)} \\ &\quad - \frac{h}{3}\left(7y' - 2y' + 2hy'' - h^2y''' + \frac{h^3}{3}y^{(4)} + y' - 2hy'' + 2h^2y''' - \frac{4h^3}{3}y^{(4)}\right) + \mathcal{O}(h^5) \\ &= \frac{1}{3}h^4y^{(4)} + \dots\end{aligned}$$

   The stability polynomial is $0 = z^2 - 1 = (z-1)(z+1)$. The root condition is satisfied, the order of accuracy is $p = 3$, and the formula is convergent.

(d) $y_{i+1} = \frac{18}{19}(y_i - y_{i-1}) + y_{i-3} + \frac{6}{19}h(f_{i+1} + 4f_i + 4f_{i-2} + f_{i-3})$. Local truncation error:

$$\begin{aligned}
\tau_{i+1} &= y(t_i + h) - \frac{18}{19}y(t_i) + \frac{18}{19}y(t_i - h) - y(t_i - 3h) \\
&\quad - \frac{6h}{19}(y'(t_i + h) + 4y'(t_i) + 4y'(t_i - 2h) + y'(t_i - 3h)) \\
&= y + hy' - \frac{18}{19}(y - y + hy') - y + 3hy' - \frac{6h}{19}(y' + 4y' + 4y' + y') + \mathcal{O}(h^2) \\
&= -\frac{2}{19}hy' + \dots
\end{aligned}$$

The stability polynomial is $0 = z^4 - \frac{18}{19}z^3 + \frac{18}{19}z^2 - 1$. Its roots are not easily found. However, the order of accuracy is $p = 0$, so the formula is not convergent.

(e) $y_{i+1} = -y_i + y_{i-1} + y_{i-2} + 2h(f_i + f_{i-1})$. Local truncation error:

$$\begin{aligned}
\tau_{i+1} &= y(t_i + h) + y(t_i) - y(t_i - h) - y(t_i - 2h) - 2h(y'(t_i) + y'(t_i - h)) \\
&= y + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' + y - y + hy' - \frac{h^2}{2}y'' + \frac{h^3}{6}y''' - y + 2hy' - \frac{4h^2}{2}y'' \\
&\quad + \frac{8h^3}{6}y''' - 2h\left(y' + y' - hy'' + \frac{h^2}{2}y'''\right) + \mathcal{O}(h^4) \\
&= \frac{2}{3}h^3y''' + \dots
\end{aligned}$$

The stability polynomial is $0 = z^3 + z^2 - z - 1 = (z-1)(z+1)^2$. The order of accuracy is $p = 2$, but the root condition is not satisfied because the root $z = -1$ is not simple. Hence the formula is not convergent.

4. (a) Applying RK2 to $y' = \lambda y$ gives $k = \lambda y_i$ and so

$$y_{i+1} = y_i + h\lambda(y_i + (h/2)\lambda y_i)$$

which is

$$y_{i+1} = (1 + h\lambda + (h\lambda)^2/2)y_i$$

The time-stability polynomial comes from substituting $y_i = Az^i$ so that

$$z = 1 + h\lambda + (h\lambda)^2/2$$

which is linear and we require $|z| < 1$ for time stability.
If λ is real and $\lambda < 0$ then $|1 + h\lambda + (h\lambda)^2/2|$ is no longer a complex modulus, and so

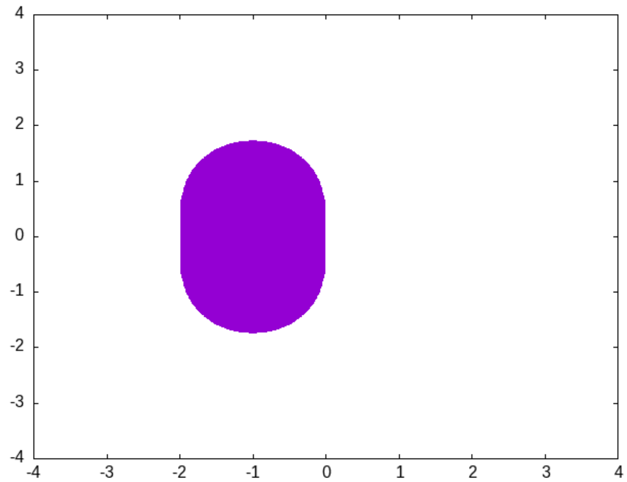
$$-1 < 1 + h\lambda + (h\lambda)^2/2 < 1$$

For the upper bound, $h\lambda + (h\lambda)^2/2 < 0$ implies $h > 0$ and $-h\lambda < 2$ (remembering $\lambda < 0$) or $h < (-2/\lambda)$. The lower bound is never violated. Thus the RK2 method is time stable for $0 < h < -2/\lambda$ for real negative λ.

(b) Here we get onto the boundary of the time stability region $|z| < 1$ by letting $z = \mathrm{e}^{i\theta}$. Thus

$$1 + \bar{h} + \bar{h}^2/2 = \mathrm{e}^{i\theta}$$

for $\bar{h} = h\lambda$. Solving the quadratic for $\bar{h}$ gives the equation requested.



The figure shows the time-stability region for RK2 in $\bar{h}$-plane.

(c) Applying the new method to $y' = \lambda y$ gives $k = \lambda y_{i+1}$

$$y_{i+1} = y_i + h\lambda(y_{i+1} - (h/2)\lambda y_{i+1})$$

So now

$$(1 - h\lambda + (h\lambda)^2/2)y_{i+1} = y_i$$

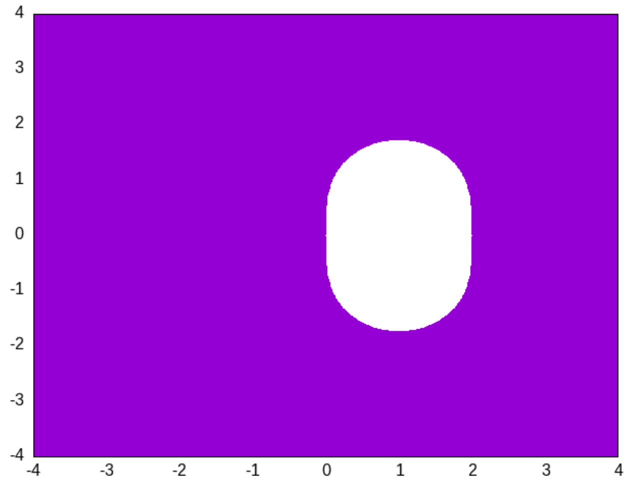
and time stability requires that

$$|1 - h\lambda + (h\lambda)^2/2| > 1$$

(Note GREATER, like backward Euler in the notes). Taking λ to be real and negative means we can intepret this as a real condition on the real line so that time stability is ensured when

$$1 - h\lambda + (h\lambda)^2/2 > 1, \qquad \text{or} \qquad 1 - h\lambda + (h\lambda)^2/2 < -1$$

but since $-h\lambda > 1$ by definition, the first condition is always met and so the method is time-stable for all h. In the complex plane the time-stability region is everywhere outside a region in the RH complex plane as in the second figure. But we are only interested in the LH plane and so this method is always time-stable.



5. We investigate the linear multistep formula

$$y_{i+1} = (1-\eta)y_i + \eta y_{i-1} + \frac{1}{2}(\eta+3)hf_i + \frac{1}{2}(\eta-1)hf_{i-1} \qquad (*)$$

which is a linear combination of AB2 ($\eta = 0$) and the central difference formula ($\eta = 1$).

(a) The zeros of the characteristic polynomial are determined by

$$0 = z^2 - (1-\eta)z - \eta = (z-1)\,(z+\eta)\,,$$

and are given by $z = 1$ and $z = -\eta$. They satisfy the root condition if $-1 < \eta \leq 1$. This is the range of η over which the formula (*) is stable.

(b) The local truncation error is

$$\begin{aligned}
\tau_{i+1} &= y(t_i+h) - (1-\eta)y(t_i) - \eta y(t_i-h) - \frac{h}{2}(\eta+3)y'(t_i) - \frac{h}{2}(\eta-1)y'(t_i-h) \\
&= y + hy' + \frac{h^2}{2}y'' + \frac{h^3}{6}y''' + \frac{h^4}{24}y^{(4)} - (1-\eta)y - \eta y + \eta hy' - \frac{\eta h^2}{2}y'' + \frac{\eta h^3}{6}y''' \\
&\quad - \frac{\eta h^4}{24}y^{(4)} - \frac{h}{2}(\eta+3)y' - \frac{h}{2}(\eta-1)\left(y' - hy'' + \frac{h^2}{2}y''' - \frac{h^3}{6}y^{(4)}\right) + \mathcal{O}(h^5) \\
&= \frac{5-\eta}{12}h^3y''' - \frac{1-\eta}{24}h^4y^{(4)} + \ldots
\end{aligned}$$

We find a local truncation error of $\mathcal{O}(h^3)$ and hence an order of accuracy $p = 2$ for all $\eta \neq 5$. The case $\eta = 5$ gives $p = 3$ but since the formula is then unstable this case is not important.

(c) The scheme starts to produce sensible answers at $h = 0.005$. We assume that the errors are then given by $\text{error}(h) \approx c\,h^p$ where c is some constant and p is the order of accuracy. To get rid of the unknown constant c we consider the quotients of the errors for different values of h.

$$\frac{\text{error}(h_1)}{\text{error}(h_2)} \approx \left(\frac{h_1}{h_2}\right)^p$$

We insert the numerical values and compare the two sides of this equation for $p = 2$. We find

$$\frac{\text{error}(0.005)}{\text{error}(0.0025)} \approx 4 \quad \text{compared to} \quad \left(\frac{0.005}{0.0025}\right)^2 = 4,$$

and

$$\frac{\text{error}(0.0025)}{\text{error}(0.001)} \approx 6.248 \quad \text{compared to} \quad \left(\frac{0.0025}{0.001}\right)^2 = 6.25.$$

We obtain good agreement and conclude that the errors behave as $\mathcal{O}(h^2)$ as expected.

(d) Note that setting $x(t) = y(t) - \cos(t)$ transforms equation (1) into $x'(t) = -100\,x(t)$. This is of the general form that is used for investigating time stability. The multistep formula for this problem with $\eta = 0.5$ has the form

$$x_{i+1} = 0.5x_i + 0.5x_{i-1} - 175hx_i + 25hx_{i-1}. \qquad (*)$$

We obtain the stability polynomial by inserting $x_i = z^i$

$$0 = z^2 - (0.5 - 175h)z - (0.5 + 25h).$$

The method is time stable for those values of h for which the two roots of the stability polynomial both have modulus $|z| < 1$. One way to continue would be to determine the two roots of the stability polynomial explicitly, but this is lengthy. One can find the borders of the time stability region also by solving the equation for h and inserting $z = 1$ and $z = -1$, respectively. This results in

$$h = \frac{z^2 - 0.5z - 0.5}{25 - 175z} = \begin{cases} 0 & \text{if} \quad z = 1, \\ 0.005 & \text{if} \quad z = -1. \end{cases}$$

We obtain the time stability region as $0 < h < 0.005$. This agrees approximately with the numerical results which are reasonable for $h \leq 0.005$.

(a) The exact solution to the homogeneous second order ODE is found by assuming solutions of the form $y(t) = A\mathrm{e}^{rt}$ and using it to derive the characteristic equation

$$r^2 + 2\gamma r + \omega^2 = 0$$

The two solutions are

$$r = -\gamma \pm \sqrt{\gamma^2 - \omega^2}$$

and so the general solution is

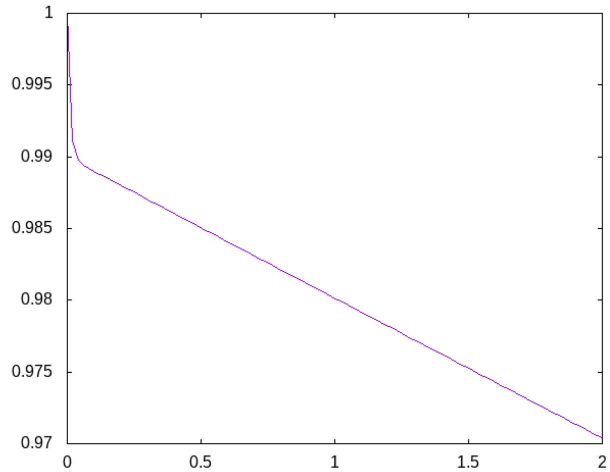
$$y(t) = A\mathrm{e}^{(-\gamma - \sqrt{\gamma^2 - \omega^2})t} + B\mathrm{e}^{(-\gamma + \sqrt{\gamma^2 - \omega^2})t}$$

[Additional note for interest: if $\gamma \gg \omega$ then we see from writing

$$-\gamma \pm \sqrt{\gamma^2 - \omega^2} = -\gamma \pm \gamma(1 - (\omega/\gamma)^2)^{1/2} \approx -\gamma \pm \gamma\left(1 - \frac{\omega^2}{2\gamma^2} + \dots\right)$$

So one root is approximately -2γ and $-\omega^2/2\gamma$ when $\omega \ll \gamma$ and so the approximate solution is

$$y(t) = A\mathrm{e}^{-2\gamma t} + B\mathrm{e}^{-(\omega^2/2\gamma)t}$$



Note that the first term is rapidly decaying and the second term is slowly decaying. Going further, application of the initial condition would give $A = -\omega^2/(4\gamma^2 - \omega^2)$, $B = 4\gamma^2/(4\gamma^2 - \omega^2)$. Then A would be small and B would be nearly equal to 1. This is interesting, since it shows that there is a rapid decay by a small amount for short times, and then a slow decay over the remaining displacement over longer times. See figure.]

(b) If $y'(t) = v(t)$ then, since $v'(t) = y''(t)$, we have from the ODE that

$$v'(t) = -2\gamma v(t) - \omega^2 y(t)$$

subject to $y(0) = 1$, $v(0) = 0$.

(c) Applying Euler to the first ODE gives

$$y_{i+1} = y_i + hv_i$$

and the second gives

$$v_{i+1} = v_i + h(-2\gamma v_i - \omega^2 y_i).$$

The initial conditions translate to $y_0 = 1$ and $v_0 = 0$.

(d) Following notes, we take the first Euler equation and write

$$v_i = (y_{i+1} - y_i)/h$$

and substitute into the second equation to give

$$(y_{i+2} - y_{i+1})/h = (y_{i+1} - y_i)/h + h(-2\gamma(y_{i+1} - y_i)/h - \omega^2 y_i).$$

which gives

$$y_{i+2} = 2y_{i+1} - y_i - 2\gamma h(y_{i+1} - y_i) - \omega^2 h^2 y_i.$$

This is a second order difference equation for y_i and we have $y_0 = 1$ and $y_1 = y_0 + hv_0 = 1$ also.

(e) Solving the difference equation in the usual way: substitute $y_i = Az^i$ to get the quadratic equation

$$z^2 - 2(1-\gamma h)z + (1 - 2\gamma h + \omega^2 h^2) = 0.$$

Its roots are

$$z = (1-\gamma h) \pm \sqrt{(1-\gamma h)^2 - (1 - 2\gamma h + \omega^2 h^2)}$$

which simplifies to

$$z = (1-\gamma h) \pm \sqrt{\gamma^2 h^2 - \omega^2 h^2}$$

and then to

$$z = 1 - h(\gamma \pm \sqrt{\gamma^2 - \omega^2}).$$

(f) For solutions which decay as $i \to \infty$ (which is required from the exact solution) we require both roots to be less than unity in modulus, i.e.

$$|1 - h(\gamma \pm \sqrt{\gamma^2 - \omega^2})| < 1.$$

Now $\gamma \pm \sqrt{\gamma^2 - \omega^2} > 0$ and $h > 0$ and so $1 - h(\gamma \pm \sqrt{\gamma^2 - \omega^2}) < 1$ is always guaranteed. However, the lower bound will be violated when

$$1 - h(\gamma \pm \sqrt{\gamma^2 - \omega^2}) < -1$$

implying

$$h > \frac{2}{\gamma \pm \sqrt{\gamma^2 - \omega^2}}.$$

Since $\gamma + \sqrt{\gamma^2 - \omega^2} > \gamma - \sqrt{\gamma^2 - \omega^2}$, the lower bound is violated when

$$h > \frac{2}{\gamma + \sqrt{\gamma^2 - \omega^2}}.$$

(g) This is a symptom of the restriction of time-stability on the Euler method, as established by application of Euler to the canonical time-stability problem. In fact, it is very closely related since the damping in the original 2nd order ODE is 2γ and, for $\gamma \gg \omega$, the method is only stable for $0 < h \lesssim 2/(2\gamma)$. In the notes we proved a similar bound $|-2/\lambda|$ where $-\lambda$ is the damping.

Note: this solution tells us that very large damping in a mechanical mass-spring-damper system requires a very small step size even though the decay of solutions is very gradual. This problem gives rise to the attribution "stiff ODE" to this type of problem since large damping implies high mechanical stiffness.