Mechanics 1: The Phase Plane for One Dimensional Motion

We have seen that a solution to Newton’s equation in one dimension with conservative forces, i.e.,

\[ m \frac{d^2 s}{dt^2} = -\nabla V(s), \quad s(t_0) = s_0, \; \dot{s}(t_0) = v_0, \tag{1} \]

must satisfy

\[ \frac{1}{2}m\dot{s}^2 + V(s) = E, \tag{2} \]

where \( E \) is a constant, called the total energy, which is determined by the initial conditions \( s(t_0) = s_0, \; \dot{s}(t_0) = v_0 \). We now want to exploit this fact and develop a graphical way of understanding all possible solutions to (1) (i.e., for any choice of initial condition). This technique is called phase plane analysis. It works for differential equations in general, and you will learn more about that later on.

First, what is this phase “plane”? After all, we are studying motion in one dimension. The first step to understanding this is a closer examination of (2) and giving it a geometrical interpretation. Let us consider (2) as a function of two variables (which it is, of course), \( s \) and \( \dot{s} \) (and, you guessed it, the phase plane will be the plane with coordinate axes \( s \) and \( \dot{s} \)). We will call this function:

\[ H(s, \dot{s}) = \frac{1}{2}m\dot{s}^2 + V(s), \tag{3} \]

where we now denote \( \dot{s} \equiv v \).

We refer to \( H(s, \dot{s}) \) as the energy function, and the interpretation we have from above is that the level sets, or level curves of the energy function are trajectories of (1) (we will return to the initial condition question shortly, but, for the moment, we will leave it as we focus on developing this geometrical picture). The level sets, or level curves of a function are the set of points in the domain of a function where the function is a constant. In our case, the level sets of \( H(s, \dot{s}) \) are the sets of points in the \( s - \dot{s} \) plane for which \( H(s, \dot{s}) = \text{constant} \). “Typically”, these level sets are curves. Let’s consider a familiar example.

**Example.** Consider the function:

\[ H(s, \dot{s}) = s^2 + \dot{s}^2. \]

The level sets of this function, i.e.,

\[ \{(s, \dot{s}) \in \mathbb{R}^2 \mid H(s, \dot{s}) = s^2 + \dot{s}^2 = E = \text{constant} \}, \]

are the circles of radius \( \sqrt{E} \). Clearly, the level sets are defined only for constants \( E \geq 0 \). They are circles (curves) for \( E > 0 \). For \( E = 0 \) the level set is a point, the origin. For certain exceptional values the level sets may degenerate to a point (which is why we put “typically” in quotes above). So now you see how to “solve” (1). Write down the energy function and plot its level sets in the \( s - \dot{s} \) plane. But in what sense does this “solve” (1)? We need to develop this geometrical picture further in order to see this.

Suppose \( s(t) \) is a solution of (1). Then it follows that \( (s(t), \dot{s}(t) \equiv v(t)) \) lies on a level set of the energy function, for some energy (constant) \( E \), i.e.,

\[ H(s(t), v(t)) = \frac{1}{2}mv(t)^2 + V(s(t)) = E. \tag{4} \]

Then we have:

\[ \frac{d}{dt} H(s(t), v(t)) = \frac{\partial H}{\partial s} \frac{ds}{dt} + \frac{\partial H}{\partial v} \frac{dv}{dt} = \nabla H \cdot \left( \frac{ds}{dt}, \frac{dv}{dt} \right) = 0. \tag{5} \]

Note that in one dimension the gradient is a scalar (as opposed to a vector in 2 or 3 dimensions) and has the much simpler form, \( \nabla V(s) \equiv \frac{dV}{ds}(s) \).
As \( t \) varies, \((s(t), v(t))\) traces out a curve in the phase plane, which is a solution of (1), and a level set of (2). Now, for any fixed \( t \), \((\frac{ds}{dt}(t), \frac{dv}{dt}(t))\) is a vector tangent to the level set. It then follows from (5) that 
\[
(\frac{\partial H}{\partial s}(s(t), v(t)), \frac{\partial H}{\partial v}(s(t), v(t))) = \nabla H(s(t), v(t))
\]
is a vector perpendicular to the level set at that point (in the sense of being perpendicular to the tangent vector to the curve at that point).

For each value of \( t \), \((\frac{ds}{dt}(t), \frac{dv}{dt}(t))\) is a vector in the \( s - v \) plane that is tangent to a trajectory of (1). It is in this sense that we say that (1) defines a vector field in the phase plane, and the solutions of (1) are curves, parametrised by \( t \), have the property that for each \( t \) the tangent vector is given by the vector field. This is made more explicit if we rewrite (1) in the form of a first order equation for the vector \((\frac{ds}{dt}(t), \frac{dv}{dt}(t))\). This is done as follows:

\[
\begin{align*}
\dot{s} &= v, \\
\ddot{s} &= \dot{v} = -\frac{1}{m}\nabla V(s).
\end{align*}
\]  

(6)

It should be clear that (6) is equivalent to (1) in the sense that a solution of (6) is equivalent to a solution of (1), and vice-versa. The geometrical picture that we have developed is illustrated in Fig. 1.

Now we describe a graphical way of drawing the level sets of the energy function in the phase plane, or “constructing the phase portrait”. We can solve (2) for \( v \) as a function of \( E \) and \( s \):

\[
v = \pm \sqrt{\frac{2}{m}} \sqrt{E - V(s)}.
\]  

(7)

For fixed \( E \), this defines the corresponding level set of the energy function as the graph of a function of \( s \). Well, “almost”, there is the \( \pm \) to consider. However, this actually makes things simpler. We must deal with both the \( + \) and \( - \) signs separately. First we consider the \( + \) sign. Since the square root of a function is positive then \( v \) will be positive. We then plot the graph of the function. Next we consider the minus sign. But it is the same graph, with just an overall minus sign. Hence, we just reflect the graph constructed with the plus sign about the horizontal axis in the phase plane (you’d better think carefully about this!).

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But still, how do we construct the graph to begin with, with the plus sign? There is a simple graphical procedure for this, which we now describe.

Plot the potential energy as a function of \( s \), and immediately below this graph draw the \( s \) and \( v \) axes of the phase plane, as shown in Fig. 2.

![Figure 2:](image)

For the potential energy function sketched in the figure, we will sketch the phase portrait. This means sketch the level sets of the energy function (2) for different values of \( E \).

Pick a value of \( E \), say \( E_1 \). Draw the horizontal line \( V(s) = E_1 \) on the graph of the potential as shown in Fig. 2. Now consider (7) with the plus sign. From this expression it should be clear that \( v \) exists only for the range of \( s \) for which \( E_1 - V(s) \geq 0 \). The values of \( s \) for which \( E_1 - V(s) = 0 \) (i.e., \( v = 0 \)) are called turning points. Start at the left hand value of \( s \) for which \( E_1 - V(s) = 0 \). Then move to the right (i.e, increase \( s \)). In this case \( v \) increases since \( E_1 - V(s) \) increases. As we move to the right, \( E_1 - V(s) \) continues to increase until it reaches a maximum value, then it decreases to zero at the right hand point where \( E_1 - V(s) = 0 \). This results in a curve in the positive \( v \) half plane. If we take the minus sign in (7), we get the negative of this curve, which results in the closed curve shown in Fig. 2. We see that the turning points are aptly named since the velocity changes direction at those points. We have also shown a direction of motion on these curves. You should convince yourself that this is correct. We can repeat this construction for a larger value of energy, say \( E_2 \) as shown in Fig. 2. We get another closed path that contains the one previously constructed with lower energy.

What about the point at the origin, which is a local minimum of the potential? At this point \( \frac{dV}{ds}(0) = 0 \), which implies that \( \dot{v} = 0 \) from (6). Also, \( v = 0 \), which implies \( \dot{s} = 0 \) from (6). Hence, the vector field is zero at this point. The point cannot “move”. It is called an equilibrium point. In general, any point for which \((s, v)\), where \( v = 0 \) and \( s \) satisfies \( \frac{dV}{ds} = 0 \) is an equilibrium point. For this example we say that it is
a stable equilibrium point since solutions starting nearby oscillate around it, i.e., they never move far away. The closed curves in Fig. 2 are examples of periodic solutions.

Now let’s consider a more complicated example, as shown in Fig. 3. What makes this example more complicated is that the potential has multiple relative extrema (sometimes called “critical points”).

First, note that there are three equilibrium points, two stable (corresponding to relative minima of the potential), and one saddle point (corresponding to a relative maximum of the potential). Following the graphical procedure described above, we obtain the level sets of the energy function for the four different values of energy shown in Fig. 3.

![Figure 3:](image)

You should be able to convince yourself that for energies $E_1$, $E_2$, and $E_4$ you obtain closed curves in the phase plane (but think about where the “dimples” in the closed curves arise from). The energy $E_3$ is slightly different. This is the energy of the saddle point, and now you will be able to see where the name comes from. At this energy a level set of the energy function passes through this point as shown in Fig. 3. If you take into consideration the direction of motion of the level set, the saddle point nature is clear, but let’s look at this point more closely.

For Figs. 2 and Fig. 3 why did we put the arrows on the curves as indicated (i.e, I’m worried about getting the direction right)? The arrows are in the direction of motion. Refer back to (6). Note that for $v > 0$, we have $\dot{s} > 0$, which means that the direction is from left to right. For $v < 0$, we have $\dot{s} < 0$, which means that the direction is from right to left. Make sure you understand this argument.

In Fig. 3 the level set of the energy function with energy $E_3$ is an example of a separatrix, i.e, a curve that separates qualitatively distinct level sets of the energy function. It is the level set that passes through a relative maximum of the potential energy.

The phase portrait is understood if we sketch the qualitatively distinct curves in the phase plane? What
do we mean by qualitatively distinct? In Fig. 2 all of the level sets of the energy function were “concentric” closed curves in the phase plane. We say that they are all qualitatively similar. Each could be smoothly deformed into any other by shrinking or growing, and without passing through any equilibria. That is not the case in Fig. 3 due to the presence of the saddle point and its associated separatrix. The separatrix defines the boundary between regions with qualitatively similar level sets of the energy function. Therefore, if there are no saddle points the phase portrait is fairly simple.

There have been a lot of ideas in this lecture. Now we summarize the main points.

**Phase Plane.** The plane with horizontal axis given by $s$, and vertical axis given by $\dot{s} = v$.

**Level Set of the Energy Function.** The set of points in the phase plane with equal energy. Typically, this is one or more curves, but could also be an isolated point. These are sometimes called *phase curves*, or *trajectories*.

**Closed Level Set of the Energy Function, not Containing any Equilibria.** These are periodic solutions of the system.

**Turning Point.** The point on the horizontal axis where $\dot{v} = 0$.

**Equilibrium Point.** A point in the phase plane where $\dot{s} = 0$ and $\dot{v} = 0$, simultaneously. Equivalently, a relative extrema of the potential energy.

**Stable Equilibrium Point.** An equilibrium point which is a relative minima of the potential energy, i.e., at the equilibrium point we have $\frac{dV}{ds} = 0$ and $\frac{d^2V}{ds^2} > 0$.

**Saddle Point.** An equilibrium point which is a relative maxima of the potential energy, i.e., at the equilibrium point we have $\frac{dV}{ds} = 0$ and $\frac{d^2V}{ds^2} < 0$.

**Phase Portrait.** An illustration of the qualitatively distinct level sets of the energy function in the phase plane.

**Separatrix.** A level set of the energy function that passes through a relative maximum of the potential energy. It separates qualitatively distinct level sets of the energy function.

**Vector Field.** The field of velocity vectors in the phase plane defined by the two-dimensional, first order form of Newton’s equations.