Mechanics 1: Vectors

Broadly speaking, mechanical systems will be described by a combination of scalar and vector quantities. A scalar is just a (real) number. For example, mass or weight is characterized by a (real and nonnegative) number. A vector is characterized by a nonnegative real number (referred to as a magnitude), and a direction. For example, momentum and force are examples of vector quantities, since it is not only the magnitude of these quantities that is important, but they also have a “directional nature” (for the moment, we are just appealing to your intuition here). We assume that you are familiar with the algebraic rules associated with manipulating numbers. Similar algebraic rules exist for manipulating vectors, and that is what we want to develop now (for example, it is important to know how to deal with several different forces acting on a “body” in different directions). However, we are getting a bit ahead of ourselves. First, we need to give a mathematical definition of a vector. The rules of vector algebra are then derived from the properties ascribed to vectors through the definition (the axiomatic way of reasoning).

As you progress in your studies of mathematics you will see that the notion of a vector is fundamental in many areas of studies. This is possible because the fundamental idea has been generalized by virtue of its being abstracted and axiomatized. However, the abstraction and axiomatization sprang from a fundamental, and fairly simple, idea (it’s always insightful to look for the motivation behind the abstraction), and this is how we will develop the definition of a vector.

The Definition of a Vector

We now define the notion of a vector that is sufficient for the purposes of this course. In other courses you will generalize the notion of a vector, but the fundamental properties will remain the same. It is important to realize that our definition of a vector does not require reference to a specific coordinate system. It is “coordinate free”. This is important to realize because later we will ”represent” vectors in different coordinate systems.

Our idea of the three dimensional space in which we live we are taking as a basic, undefined concept. We consider two points in space, labelled $P$ and $Q$, and the line segment connecting them, starting at $P$ and ending at $Q$ see Fig. 1.

![Figure 1: The definition of a vector as a directed line segment between the points $P$ and $Q.$](image)

Hence, a vector is a directed line segment, in the sense that its direction is from $P$ to $Q$. This is denoted in the figure by the arrow on the line segment. The magnitude of the vector is just the length of the vector.

Our definition of a vector is valid in ordinary (three-dimensional) space. Do not let the fact that the paper on which Fig. 1 is drawn fool you. Work out in your mind that the definition is perfectly valid in
three dimensions. Geometrical reasoning is very powerful in mathematics. Many complicated ideas can be conveyed (and, allegedly, understood) with a simple sketch. However, if the limitations of the sketch are not well understood from the beginning, then false ideas can also be conveyed. There is a bit of art work involved here; something that is gained with experience and careful thought. Some mathematicians may give you the impression that mathematics is a very “cut-and-dried” subject; that it flows logically through an orderly current of successive theorems, proofs, lemmas, propositions, and definitions. This is nonsense! Many important areas of mathematics were originally formulated in a very confused, illogical, and messy state, and it’s only through many years of thought and refinement that they are brought into the perfectly crisp and logical state that they are presented in today (which is also often dry and uninspiring).

Now we need to introduce some notation for vectors. That is, a way of representing them on a piece of paper, or a black or whiteboard. This is important. Notation forms the language of mathematics, so you must get this right from the start. In the history of mathematics many battles have been fought over the best notation for a subject (a notable such battle was the one between Newton and Leibniz (and their respective followers) for the notation of the calculus that you are learning this year).

The vector shown in Fig. 1 can be denoted in several ways:

$$\vec{PQ}, \vec{A}, \vec{A}.$$ 

The first notation, $$\vec{PQ}$$, probably makes a lot of sense from the point of view of the definition of a vector that we gave (and good mathematical notation should be intuitive in the sense that it conveys the aspects of the quantity that is being represented). Unfortunately, we will rarely use such notation. Now you are probably wondering where the “$$A$$” came from. It’s certainly not shown in Fig. 1. It is at this point that we perform a bit of abstraction with respect to our original definition of a vector. A vector is defined by a “length and direction”. So the beginning point ($$P$$ in the figure) and the end point ($$Q$$ in the figure) are not crucial for defining a vector; any two points will do provided that the segment between those two points has the same length and direction. Or, to put it another way, we can move the line segment all over space and, so long as we maintain its length and direction, it always defines the same vector (this will be important shortly when we talk about addition of two vectors). This makes sense because, thinking ahead, we want to use vectors to represent quantities such as momentum and force (yes, things we have not defined yet). The force acting on a body has a characteristic that is independent of the particular location in space that it is being applied. For example, you can apply the same force to lifting an object whether that object is in Bristol or in London. But back to notation. Since the two particular endpoints are not essential we economize the notation and use a single character, in this case “$$A$$” to denote a vector. On the printed page it is denoted in boldface, $$\mathbf{A}$$. This doesn’t work so well on a black or whiteboard. In that case either an underline is used, $$\underline{A}$$, or an “overarrow”, $$\vec{A}$$.

Key point: A vector has two characteristics: length (or “magnitude”) and direction.

Vector Algebra

Consistent with our definition of a vector, we now define how vectors are added, multiplied by scalars (“numbers”), and subtracted.

With the definition of a vector in hand we can now develop rules for adding vectors, subtracting vectors, and multiplying vectors by scalars (the idea of multiplying two vectors will be considered a bit later). But first, we need to establish two important properties.

Equality of Two Vectors. Given two vectors, say $$\mathbf{A}$$ and $$\mathbf{B}$$, what does it mean for these two vectors to be equal. Since a vector is defined by its direction and length, we say that $$\mathbf{A}$$ and $$\mathbf{B}$$ are equal if they have the same direction and length, and we write this as $$\mathbf{A} = \mathbf{B}$$. Note that equality of two vectors does not depend at all on their beginning and end points in space.
The Negative of a Vector. Consider a vector $\mathbf{A}$. The vector having the same length as $\mathbf{A}$, but with opposite direction, is denoted by $-\mathbf{A}$.

The Sum of Two Vectors. Now let’s move on to the subject of adding two vectors. Suppose we are given two vectors $\mathbf{A}$ and $\mathbf{B}$. What meaning could we give to the sum of $\mathbf{A}$ and $\mathbf{B}$? We first place the beginning point of $\mathbf{B}$ at the ending point of $\mathbf{A}$, as shown in Fig. 2b. The the sum of $\mathbf{A}$ and $\mathbf{B}$, denoted $\mathbf{A} + \mathbf{B}$, is defined to be the vector, which we will call $\mathbf{C}$, from the beginning point of $\mathbf{A}$ to the ending point of $\mathbf{B}$, see Fig. 2b. Sometimes the word resultant is used for sum. This definition is equivalent to the parallelogram law for vector addition, which we illustrate in Fig. 2c. It should be clear that this definition for vector addition can be easily extended to more that two vectors. We illustrate this in Fig. 3.

![The Negative of a Vector](image1)

![The Sum of Two Vectors](image2)

Figure 2: a) Two vectors $\mathbf{A}$ and $\mathbf{B}$. b) The sum of $\mathbf{A}$ and $\mathbf{B}$. c) Illustration of the parallelogram law for vector addition.
The Difference of Two Vectors. The difference of two vectors $\mathbf{A}$ and $\mathbf{B}$, denoted $\mathbf{A} - \mathbf{B}$, is the vector $\mathbf{C}$ which when added to $\mathbf{B}$ gives $\mathbf{A}$. This definition, along with the definition of the negative of a vector given above, implies that we can write $\mathbf{A} - \mathbf{B}$ as $\mathbf{A} + (-\mathbf{B})$ (it would be very instructive if you drew a “picture” of this definition using the definitions given already). If $\mathbf{A} = \mathbf{B}$ then $\mathbf{A} - \mathbf{B}$ is defined as the zero vector, and denoted by 0. The length of the zero vector is zero, but its direction is undefined.

The Product of a Vector and a Scalar. The product of a vector $\mathbf{A}$ with a scalar $c$ is the vector $c\mathbf{A}$ (or $\mathbf{A}c$) with magnitude $|c|$ times the magnitude of $\mathbf{A}$ and the direction the same as $\mathbf{A}$ if $c$ is positive, or the direction opposite to $\mathbf{A}$ if $c$ is negative. If $c = 0$ then $c\mathbf{A} = 0$, the zero vector.

Unit Vector. This is an important notion that will arise many times. For this reason we choose to highlight the idea. A unit vector is simply a vector having length one (“unit length”). Given any vector $\mathbf{A}$, with length $A > 0$, we can construct a unit vector from it, as follows:

$$\frac{\mathbf{A}}{A} \equiv \mathbf{a},$$

then we can also write

$$\mathbf{A} = A\mathbf{a}.$$

For a given vector $\mathbf{A}$ we will often denote its magnitude by:

$$A = |\mathbf{A}|.$$

Note that the only thing distinguishing two unit vectors is their respective directions.

**Key point:** Vectors are added using the “parallelogram law”. This rule for addition of vectors does not require coordinates.
Kinematics–Multiplication of Vectors

There are two ways to multiply two vectors: the dot (or scalar) product and the cross (or vector product).

Now we come to the subject of multiplication of vectors. If you think of our purely geometrical definition of a vector given previously, it may seem rather strange to think of multiplying two “arrows in space”. However, our notion of multiplication will also be based on geometry.

There are two ways in which we will multiply vectors. They arise naturally in different settings in mechanics. They are the dot, or scalar product and the cross, or vector product. This should already seem a bit novel. After all, there is only one way to multiply numbers, and when you multiply two numbers, you get a number. The terminology here seems to indicate that, besides there being two ways to multiply vectors, if you do it one way you get a number and if you do it the other way you get a vector. This is indeed the case, but let’s get to the definitions.

Dot, or Scalar Product. Let \( \mathbf{A} \) and \( \mathbf{B} \) be vectors. Then the dot, or scalar product of \( \mathbf{A} \) and \( \mathbf{B} \), denoted \( \mathbf{A} \cdot \mathbf{B} \), is defined by:

| Key point: |
| \[ \mathbf{A} \cdot \mathbf{B} \equiv |\mathbf{A}| |\mathbf{B}| \cos \theta, \quad 0 \leq \theta \leq \pi, \] |

where \( \theta \) is the angle between \( \mathbf{A} \) and \( \mathbf{B} \) when we give \( \mathbf{A} \) and \( \mathbf{B} \) the same starting points, see Fig. 4.

![Figure 4](image)

We say that two vectors are perpendicular, if their dot product is zero. An equivalent term is “orthogonal”. Two vectors are said to be orthogonal if their dot product is zero. A related term is “orthonormal”, but this tends to be applied exclusively to unit vectors. Two unit vectors are said to be orthonormal if their dot product is zero.

We noted earlier that we will assume certain basic knowledge and take some ideas as evident, e.g. the notions of space, time, and mass. Concerning space, the notion that “two points determine a straight line”, is also something that most of you should accept based on your experience with Euclidean geometry. In the hierarchy of Euclidean geometrical objects, the next most complicated object (beyond a line) would be a plane. You should understand completely, and make sure you are comfortable with, the statements “three points determine a plane” and “two vectors determine a plane”. If you have doubts about these statements, bring them up in your tutorial.

With the dot product of two vectors defined we can discuss the notion of “the projection of one vector on another”.

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The Projection of a Vector onto Another Vector. Consider the vectors $A$ and $B$ as shown in Fig. 5. What do we mean by “the projection of $A$ on $B$?”

Figure 5:

Let $b = \frac{B}{|B|}$ denote the unit vector in the direction of $B$. Then the projection of $A$ on $B$ is given by:

$$A \cdot b = |A| \cos \theta.$$  

Why didn’t we define the projection of $A$ on $B$ as just $A \cdot B$?

Cross, or Vector Product. Let $A$ and $B$ be vectors. Then the cross, or vector product of $A$ and $B$, denoted $A \times B$, is defined by:

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where $\theta$ is the angle between $A$ and $B$ and $\mathbf{n}$ is a unit vector perpendicular to the plane formed by $A$ and $B$.

However, $\mathbf{n}$ is not uniquely defined in this way. There are two possibilities, and we will have to make a choice of one so that we are all speaking the same language.

Here’s how to understand the possibilities. Take your right hand, and point the thumb in the direction of $A$ with the fingers in the direction of $B$, when you do this consider Fig. 6. Then $\mathbf{n}$ could be chosen to either point “upward” out of your hand (Fig. 6a), or “downward”, through your hand (Fig. 6b). We make the first choice, i.e. that shown in Fig. 6a.
If \( \mathbf{A} = \mathbf{B} \), or \( \mathbf{A} \) and \( \mathbf{B} \) are collinear (parallel), then we define \( \mathbf{A} \times \mathbf{B} = 0 \), the zero vector.

**A Final Remark: Dimensions.** We have been taking the notions of two and three dimensional space as primitive concepts since most people’s experience makes them comfortable with these notions. However, many applications of mechanics involve the consideration of higher dimensional space. How do we define vectors in higher dimensions? The graphical constructions we’ve utilized here don’t appear to generalize easily. This is where *abstraction* comes in, and you will learn about it in detail later in your studies.

**Key point:**
- The dot product of two vectors is a scalar.
- The cross product of two vectors is a vector.
- The definitions of dot product and cross product do not require coordinates.